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# One-dimensional field theories arising from continuous measurement

## Master Thesis

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by

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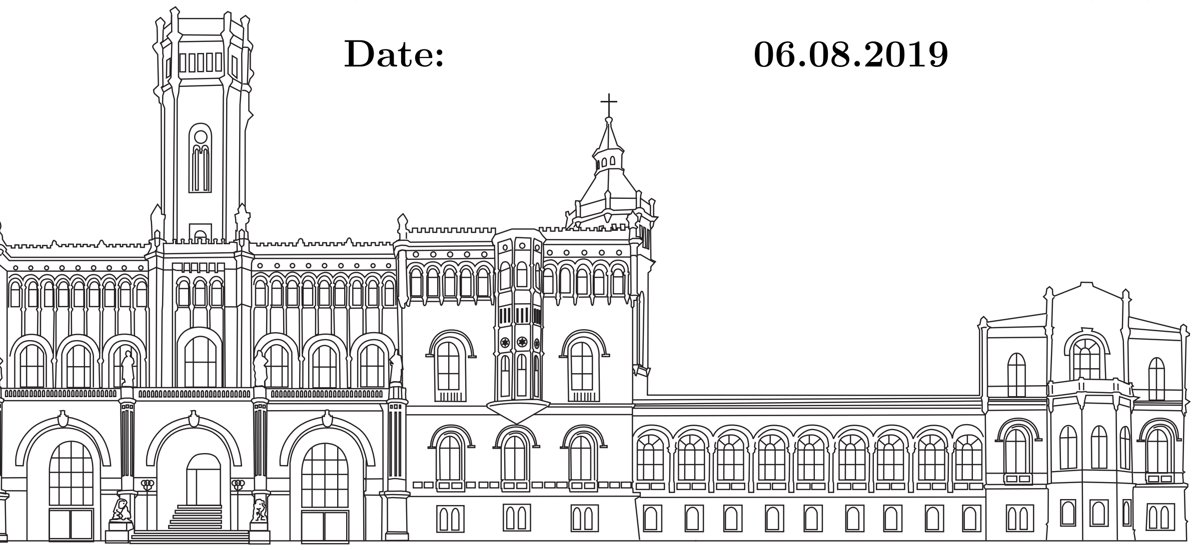
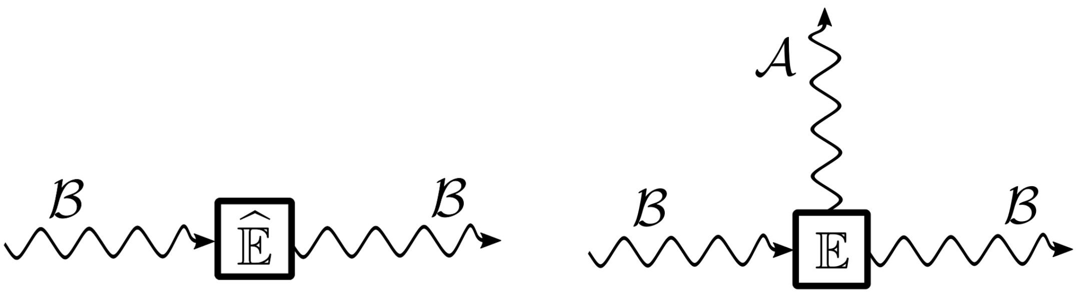


# Contents

# Chapter 1

# Outline

This thesis analyzes the possibility of taking mathematical rigorous limits of measurements on certain quantum mechanical systems and discusses necessary criteria for this limit to be well defined. Since the standard formalism of quantum mechanics is not capable of capturing those elaborate limit constructions, due to its lack of mathematical generality, we will always see quantum mechanics from the "quantum information" viewpoint, i.e. describing time evolution of states via quantum channels $\widehat{\mathbb{E}}$ between certain C$^*$-algebras $\mathcal{B}$. For the sake of presentation we will display this graphically in the following way:




(a) A quantum channel

(b) A quantum channel with dilation

The second figure already used Stinesprings factorization theorem to construct a so-called "dilation space" corresponding to the C$^*$-algebra $\mathcal{A}$. Both channels $\widehat{\mathbb{E}}$ and $\mathbb{E}$ are equivalent. After taking the limit, those channels induce non trivial dynamics on the dilation space, which we will view as a one-dimensional quantum field. The typical example for such a system would be an atom in a cavity. The out-coupled light field would then correspond to the quantum field described above.
We end by briefly discussing properties of this field and give some outlooks about further interesting research topics involving this construction. Let us briefly summarize the content of each chapter.

In chapter 2 we collect the preliminaries we will need out of a variety of different mathematical subjects. We use this chapter to fix notation and to give basic insights in fields like nets, inductive limits or evolutions systems, which are not part of every physicists standard literature.

Chapter 3 is used to define the basic algebraic objects needed for our limit, i.e. we define a “limit Hilbert space”, show its well definedness and discuss basic/canonical properties of this space. In analogy to quantum stochastic calculus we introduce a dense subset of this space, known as exponential vectors and discuss similarities to the Fock spaces.

Since we want to apply our formalism to some quantum states, we'll need to come up with some examples, which will be done in chapter 4. The “Finitely Correlated States” (FCS, aka MPS) can be motivated to be a fruitful choice regarding this task. Since MPS already have a continuous analogue, i.e. “cMPS”, which are defined rather heuristically, we are able to compare the two limit results. We therefore recall the first definitions of all those notions and describe how to view them in the context of continuous measurements.

In chapter 5 we will define “discrete” field operators and show their well definedness in the refinement limit. It will be more convenient for this thesis, however, to deal with Weyl operators, i.e. the objects generated by those field operators. We also proof that the Weyl operators fulfill a special form of the Weyl CCR.

Chapter 6 begins with a quick recap of second quantization and what this correspond to in the context of continuous measurements. One then defines point processes and characteristic functions which model experimental setups in typical quantum optics laboratories. Those point processes are then related to expectation values of the Weyl operators defined prior and analyzed in the limit.

The last chapter is a summary of continuous Stinespring dilations and finally defines cMPS in a new way. We analyze this notion with multiple tools and explicitly calculate an example. We end by given an outlook over possible further research topics.

# Chapter 2

# Mathematical and Physical Basics

First we want to set some notation and preliminaries. A scalar product is defined to be antilinear in the first argument and linear in the second argument and written in the Dirac formalism as Bra's and Ket's, throughout this thesis. All Hilbert spaces we encounter are implicitly defined to be complex and separable. Since this thesis needs a lot of different mathematical tools, we need to define some basic concepts out of a variety of mathematical subjects.

## 2.1 Preliminaries in Functional Analysis

The modern formalism of quantum mechanics is given by so-called "channels" between C*-algebras. We start by collecting some basic properties and definitions.

**Definition 2.1.** *Let* $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ *be normed spaces and* $A \in L(X,Y)$ *a linear operator. Denote the* ***operator norm*** *of* $A \in L(X,Y)$ *as*

$$\|A\|_{op} = \sup_{\|x\|_X \leq 1} \|Ax\|_Y. \tag{2.1}$$

*Whenever the underlying spaces are clear we will omit the indices. If* $\|A\|_{op} < \infty$ *we call* $A$ *a* ***bounded operator*** *and denote* $A \in \mathcal{B}(X,Y)$*. If, however, we will encounter some unbounded operators, we will explicitly highlight those and restrict its domain, whenever possible, to a dense subspace.*

**Definition 2.2.** *An* ***associative algebra*** *is a vector space* $\mathcal{A}$ *over* $\mathbb{C}$ *with an associative and distributive product, i.e. for all* $A, B, C \in \mathcal{A}$ *we have* *i)* $A(BC) = (AB)C$,

*ii)* $A(B + C) = AB + AC$ *and* $(A + B)C = AC + BC$,

*iii)* $(\mu A)(\nu B) = \mu\nu(AB)$ *with* $\mu, \nu \in \mathbb{C}$.

*An* ***involution*** *is a map* $\mathcal{A} \ni A \longmapsto A^\dagger \in \mathcal{A}$ *satisfying*

*i)* $A^{\dagger\dagger} = A$,

*ii)* $(AB)^\dagger = B^\dagger A^\dagger$,

*iii)* $(\mu A + \nu B)^\dagger = \mu^* A^\dagger + \nu^* B^\dagger$ *with* $\mu, \nu \in \mathbb{C}$,

*where* $\mu^*$ *denotes the complex conjugate of* $\mu \in \mathbb{C}$.

*An associative algebra with an involution is called* $^*$***-algebra*** *or* ***involutive Algebra****.*[1] *The algebra* $\mathcal{A}$ *is called* ***normed*** *with norm* $\|\cdot\| : \mathcal{A} \longrightarrow [0, \infty)$ *if*

*i)* $\|A\| = 0$ *if and only if* $A = 0$,

*ii)* $\|\mu A\| = |\mu|\|A\|$ *with* $\mu \in \mathbb{C}$,

*iii)* $\|A + B\| \leq \|A\| + \|B\|$,

*iv)* $\|AB\| \leq \|A\|\|B\|$.

Having a norm and an involution on our underlying algebra it makes sense to ask how both notions interact.

**Definition 2.3.** *Let* $(\mathcal{A}, \|\cdot\|)$ *be a normed algebra.*

*i) If* $\mathcal{A}$ *is complete w.r.t the norm topology, one calls* $\mathcal{A}$ *a* ***Banach algebra****.*

*ii) A Banach algebra* $\mathcal{A}$ *with an involution* $\dagger$*, such that* $\|A\| = \left\|A^\dagger\right\| \quad \forall A \in \mathcal{A}$ *holds, is called* $\boldsymbol{B^*}$***-algebra****.*

*iii) A* $B^*$*-algebra for which* $\left\|AA^\dagger\right\| = \|A\|^2\ \forall A \in \mathcal{A}$ *is true is called* $\boldsymbol{C^*}$***-algebra****.*

[1] Even though it is called $^*$-algebra, we will use $* = \dagger$, since the star is already used for complex conjugation.

*All those algebras canonically carry some more algebraic structures. We denote $\mathcal{A}'$ for the **topological dual space** of $\mathcal{A}$. The **set of positive elements** in $\mathcal{A}$ is defined as*

$$\mathcal{A}_{pos} := \Big\{ A \in \mathcal{A} \;\Big|\; A = A^{\dagger} \text{ and } \sigma(A) \subset [0,\infty)] \Big\},$$

*where $\sigma(A) := \big\{ \lambda \in \mathbb{C} \;\big|\; \lambda \mathbb{1}_{\mathcal{A}} - A$ is not invertible$\big\}$ is the spectrum of $A$.*

*If $A \in \mathcal{A}_{pos}$ one writes $A \geq 0$. A functional $\varphi \in \mathcal{A}'$ is said to be **positive** if $\varphi(A^{\dagger}A) \geq 0 \;\forall\, A \in \mathcal{A}$. If $\varphi$ is positive and furthermore $\|\varphi\| = 1$ holds, we call $\varphi$ a **state** and denote the set of states as $\mathcal{S}(\mathcal{A})$.*

Having those basic notions, one is able to construct the following function spaces, as it is usually done in basic functional analysis courses.

**Notation 2.4.** *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}, \mathcal{B}$ be $C^*$-algebras. The following spaces will be used frequently in the upcoming analysis:*

*i) The space of compact operators $K(\mathcal{H})$.*

*ii) The space of trace class operators $\mathfrak{T}(\mathcal{H})$.*

*iii) The Hilbert space of Hilbert-Schmidt-operators $(HS(\mathcal{H}), \langle \cdot | \cdot \rangle_{HS})$ with scalar product*

$$\langle \cdot | \cdot \rangle_{HS} : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}, \quad \langle A|B \rangle_{HS} = \operatorname{tr}\Big(B^{\dagger}A\Big).$$

*iv) The space of completely positive maps $\mathfrak{CP}(\mathcal{A}, \mathcal{B})$.*

We will encounter symplectic vector spaces, most importantly the phase space, and exploit its algebraic structure. However, it is sometimes practical to think of symplectic forms being induced by a complex structure. We will discuss this relation in the following definitions.

**Definition 2.5.** *Let $\Xi$ be a $2N$-dimensional vector space with a **symplectic form**, i.e. a bilinear, antisymmetric and non-degenerate map $\sigma : \Xi \times \Xi \longrightarrow \mathbb{R}$. $(\Xi, \sigma)$ is then called a **symplectic space**.*

This symplectic space structure helps in classical mechanics to simplify the equations of motion and to analyze the geometry of the phase space. If, however, we have a complex vector space (or a complexification of a real vector space) the symplectic structure comes with it canonically in the following way.

**Definition 2.6.** *Let $V$ be a real vector space of dimension $2N$. A real linear endomorphism $J : V \longrightarrow V$ satisfying $J^2 = -\mathbb{1}_V$ is called a* ***complex structure****.*

*If such a map is defined on all of $V$ one can define $V$ to be an $N$-dimensional complex vector space by setting $(x + iy)v := xv + yJ(v)$. One says the vector space $(V, J) =: V_{\mathbb{C}}$ is the* ***complexification*** *of $V$.*

The correspondence to symplectic vector spaces is then given by the following observation. Let $V_{\mathbb{C}}$ be the complexification of $V$, then $(V, \sigma)$ with $\sigma(x, y) = \mathrm{Im}(\langle x|y\rangle)$ defines a symplectic vector space. Showing that $\sigma$ is indeed a symplectic form is a straightforward calculation.

## 2.2 Inductive Limits

Defining limits of algebraic objects in a mathematical rigorous way is an important topic of this thesis. The most general discussion of these objects is obtained in category theory. In this thesis, however, we do not need such an abstract definition and rely on the construction of projective and inductive limits in most topology textbooks. The construction and notation in this section is adapted from [Bou04].

**Definition 2.7.** *A* ***preordered set*** *is a set $I$ with a binary relation $\leq$, that is reflexive and transitive, i.e. for all $a, b, c \in I$ we have that*

*i)* $a \leq a$ *(reflexivity)*

*ii)* $a \leq b \ \wedge \ b \leq c \Longrightarrow a \leq c$ *(transitivity).*

This definition alone is not enough to define limits properly, so one also requires:

**Definition 2.8.** *A* ***directed set*** *$I$ is a preordered set such that every finite subset of $I$ has an upper bound. Let $X$ be a topological space with topology $\mathcal{T}$, then a map $f : (I, \leq) \longrightarrow (X, \mathcal{T})$ is called a* ***net****.*

Even though, in abuse of notation, one mostly refers to an element $f_\alpha$ for $\alpha \in I$ as the net; since all examples of nets will be quite intuitive, this will be always clear from context. Since nets are a generalization of sequences it seems natural to lift the usual definition of a sequence being convergent or Cauchy to this notion. In order to do so, we restrict the image space of the nets to normed spaces. One could, however, require less structure, for example uniform spaces, but we stay with normed spaces for simplicity.

**Definition 2.9.** *A net $\eta : (I, \leq) \longrightarrow (X, \|\cdot\|), i \longmapsto \eta_i$ into a normed space $X$ is called a **Cauchy net** iff*

$$\forall \varepsilon > 0 \; \exists i \in I \; s.t. \; \forall j, k \in I, \; j, k \geq i : \left\|\eta_j - \eta_k\right\| < \varepsilon. \tag{2.2}$$

*Furthermore, $\eta_i$ is said to be a **convergent net** with limit $\tilde{\eta} =: \varinjlim_{i \in I} \eta_i \in X$ iff*

$$\forall \varepsilon > 0 \; \exists i \in I \; s.t. \; \forall j \in I, \; j \geq i : \left\|\tilde{\eta} - \eta_j\right\| < \varepsilon. \tag{2.3}$$

Since most of the nets we are dealing with a mapping into, at least, Banach spaces, the convergence of Cauchy nets becomes easy to prove. We will take advantage of this fact later on.

Given this notion, we can extend this convergence construction from elements of some normed space to more abstract categories, i.e. we can define limits of certain algebraic objects like groups or rings. However, when defining convergence in any way one needs to compare different objects with each other. In topological spaces this is done by looking whether different elements are in open neighborhoods of each other or not in contrast to normed spaces, where the norm quantifies this comparison.

Dealing with more abstract algebraic objects the role of comparing different elements needs to be lifted to homomorphisms of that underlying structure. This is the basic idea of the so-called "inductive limit". In this thesis mostly limits of vector spaces will be used, the definition is, nonetheless, more general:

**Definition 2.10.** *Let $(I, \leq)$ be a directed set and let $\left\{X_i \,\middle|\, i \in I\right\}$ be a family of algebraic objects indexed by $I$. Furthermore, let $f_{ij} : X_i \longrightarrow X_j$ be a homomorphism of the algebraic structure for every $i \leq j$ such that:*

*i)* $f_{ii}$ *is the identity of* $X_i$ *for all* $i \in I$

*ii)* $f_{ik} = f_{jk} \circ f_{ij}$ *for all* $i \leq j \leq k \in I$.

*The pair $(X_i, f_{ij})$ is called an **inductive system** over $I$. The **inductive limit** of the system $(X_i, f_{ij})$ is then defined as*

$$f\text{-}\varinjlim_{i \in I} X_i := \bigcup_i X_i \big/ \sim$$

*where* $x_i \sim x_j$ *with* $x_i \in X_i$ *and* $x_j \in X_j :\Longleftrightarrow \exists\, k \in I$ *s.t.* $f_{ik}(x_i) = f_{jk}(x_j) \in X_k$.

There is another, closely related, kind of limit space we will present here for the sake of completeness.

**Definition 2.11.** *Let* $(X_i, f_{ij})$ *be a direct system again. The* ***inverse limit*** *of this system is then defined as a subspace of the direct product of the* $\left\{X_i \,\middle|\, i \in I\right\}$ *via*

$$f\text{-}\varprojlim_{i\in I} X_i := \left\{ \boldsymbol{x} = (x_1, x_2, \dots) \in \prod_{i\in I} X_i \,\middle|\, x_i = f_{ij}(x_j)\ \forall\ i \leq j \in I \right\}. \tag{2.4}$$

Both notions are related to each other by duality. This is meant in the sense that $\mathrm{Hom}\left(\varinjlim X_i, Y\right) = \varprojlim \mathrm{Hom}\,(X_i, Y)$, so if the algebraic objects carry a vector space structure, those limits are "dual" to each other. If the family of spaces are Hilbert spaces, then the limit space will canonically carry the structure of a Pre-Hilbert space.

**Notation:** Let us fix a directed set $(I, \leq)$ and note the following subtle difference in notation.

i) If we consider a net, i.e. a map from $(I, \leq)$ with values in a topological (mostly even normed) space, then we will write $\varinjlim_{i\in I}$ for the limit.

ii) In contrast to this we can take the limit of an inductive system $(X_i, f_{ij})$. Here we would use the limit notation $\underrightarrow{f\text{-}\lim}_{i\in I}$.

So the existence of a homomorphism like $f$ in the limit notation automatically shows the reader what kind of construction was used. In abuse of language we will always refer to maps, indexed by a directed set, as "nets".

The directed set we are going to use is the following set of interval decompositions.

**Definition 2.12.** *Let* $[0, T]$ *be a fixed interval. An* ***interval decomposition*** $\Theta$ *is a finite set of points in* $[0, T]$ *where 0 and* $T$ *are always included, i.e.*

$$\Theta := \{0 = t_0 < t_1 < \cdots < t_n = T \,|\, t_i \in [0, T], n \in \mathbb{N}\}\,. \tag{2.5}$$

*The set of interval decompositions of* $[0,T]$ *will be denoted by* $\mathfrak{Z}([0,T])$ *and the set of labels (without the 0), i.e.* $\{1,\dots,n\}$*, by* $I(\Theta)$*.*

*The length of the i-th subinterval will be written as* $\tau_i = t_i - t_{i-1}$ *for* $i \in I(\Theta)$ *and number of subintervals will be denoted as* $\#I(\Theta)$*. When* $\Theta \subseteq \Xi$ *we say that* $\Theta$ ***is coarser than*** $\Xi$ *or, equivalently, that* $\Xi$ ***is finer than*** $\Theta$*.*

Those interval decompositions constitute a directed set, as it was already mentioned. There is, however, even more structure on this set.

**Lemma 2.13.** *The set of interval decompositions with the set theoretic inclusion* $\subseteq$ *defines a directed set. Even more so we have that for every* $\Theta, \Xi \in \mathfrak{Z}([0,T])$ *an upper bound is given by* $\Theta \cup \Xi$*.*

*Proof.* The proof is straightforward application of set theory and shall be omitted here. $\square$

Since this construction seems quite abstract it helps to visualize some interval decompositions. The following graphic makes the order relation between different interval decompositions intuitive.

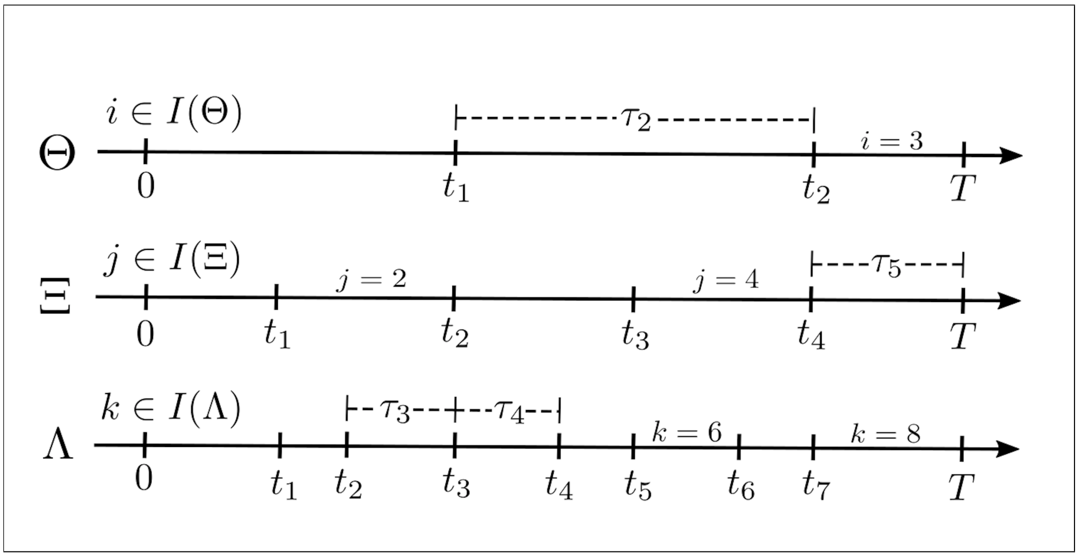



Figure 2.1: Comparison of different interval decompositions $\Theta \subseteq \Xi \subseteq \Lambda$

## 2.3 Maps Between Inductive Limit Spaces

We are mainly interested in maps from a fixed Hilbert space $\mathcal{H}$ into an inductive limit space $\mathcal{K}$, corresponding to an inductive system of Hilbert spaces $(\mathcal{K}_\Theta, J_{\Theta\Xi})$ or in maps between inductive limit spaces.[2] Since we will define operators on the discrete spaces $\mathcal{K}_\Theta$ first and then take the limit we need a notion of convergence of operators between different spaces, i.e. we need to say what it means for an operator $O_{\Theta\Xi} : \mathcal{K}_\Theta \longrightarrow \mathcal{K}_\Xi$ to converge to an operator $O : \mathcal{K} \longrightarrow \mathcal{K}$ w.r.t. the inductive system $J_{\Xi\Theta} : \mathcal{K}_\Theta \longrightarrow \mathcal{K}_\Xi$. As usual for operator convergence, we will be able to define multiple topologies corresponding to this construction.

Since, for example, the maps $O_\Theta : \mathcal{K}_\Theta \longrightarrow \mathcal{K}_\Theta$, or $V_\Theta : \mathcal{H} \longrightarrow \mathcal{K}_\Theta$ do not map into the same space when comparing them with another interval decomposition $\Xi$, it makes no sense to talk about net convergence in the first place. We will define what it means for those "nets" of operators to converge, even though they do not constitute nets in the usual sense. More formally:

**Definition 2.14.** *Let $\mathcal{H}$ be a Hilbert space and $(\mathcal{K}_\Theta, J_{\Xi\Theta})$ an inductive system of Hilbert spaces with $\mathcal{K}$ denoting the closure of the limit space and $J_\Theta : \mathcal{K}_\Theta \longrightarrow \mathcal{K}$ is the natural embedding into the limit space.*

*In abuse of notation and language we say that $V_\Theta : \mathcal{H} \longrightarrow \mathcal{K}_\Theta$ is a net of operators. We denote the (real) net of operators mapping into the limit space and indexed by $\mathfrak{Z}([0,T])$ as $\widetilde{V}_\Theta : \mathcal{H} \longrightarrow \mathcal{K}$, with $\widetilde{V}_\Theta := J_\Theta V_\Theta$ and say that $\Theta \longmapsto V_\Theta$ converges*

***in norm**: iff $\Theta \longmapsto \widetilde{V}_\Theta$ converges w.r.t. the operator norm $\|\cdot\|_{op}$ on $\mathcal{B}(\mathcal{H},\mathcal{K})$.*

***strongly**: iff $\Theta \longmapsto \widetilde{V}_\Theta |\varphi\rangle$ converges on $(\mathcal{K}, \|\cdot\|_\mathcal{K})$ for every $|\varphi\rangle \in \mathcal{H}$.*

***weakly**: iff $\Theta \longmapsto \left\langle \lambda \middle| \widetilde{V}_\Theta \varphi \right\rangle$ converges on $(\mathbb{C}, |\cdot|)$ for every $|\varphi\rangle \in \mathcal{H}$ and $|\lambda\rangle \in \mathcal{K}$.*

*Analogously we say that $O_\Theta : \mathcal{K}_\Theta \longrightarrow \mathcal{K}_\Theta$ is a net of operators.*
*Define $\widetilde{O}_\Theta : \mathcal{K} \longrightarrow \mathcal{K}$ to be $\widetilde{O}_\Theta = J_\Theta O_\Theta J_\Theta^\dagger$ and say that $\Theta \longmapsto O_\Theta$ converges*

***in norm**: iff $\Theta \longmapsto \widetilde{O}_\Theta$ converges in $(\mathcal{B}(\mathcal{K}), \|\cdot\|_{op})$.*

***strongly**: iff $\Theta \longmapsto \widetilde{O}_\Theta |\varphi\rangle$ converges on $(\mathcal{K}, \|\cdot\|_\mathcal{K})$ for every $|\varphi\rangle \in \mathcal{K}$.*

***weak-***: iff $\Theta \longmapsto \operatorname{tr}\left(\rho\, \widetilde{O}_\Theta\right)$ converges on $(\mathbb{C}, |\cdot|)$ for every $\rho \in \mathfrak{T}(\mathcal{K})$.*

[2]Note that we implicitly mean the norm closure of the inductive limit, since the limit of Hilbert spaces would only be a Pre-Hilbert space.

We can simplify this convergence properties using a variety of different notions. Firstly, dealing mostly with Banach space valued nets, we can shorten our notation by defining the following limit.

**Definition 2.15.** *Let $c_{\Xi\Theta} \in \mathbb{R}$ be a real number for every $\Xi, \Theta \in \mathfrak{Z}([0,T])$. We define a* ***discrete comparison limit*** *to be*

$$\lim_{\Xi \gg \Theta} c_{\Xi\Theta} := \varinjlim_{\Theta \in \mathfrak{Z}([0,T])} \limsup \{ c_{\Xi\Theta} \mid \Xi \in \mathfrak{Z}([0,T]), \Xi \supseteq \Theta \} . \tag{2.6}$$

Having defined convergence of nets in equation (2.3) one can simplify this, using the definition above, in the following way.

**Corollary 2.16.** *Let $\eta : (\mathfrak{Z}([0,T]), \subseteq) \longrightarrow (X, \|\cdot\|)$ a Banach space valued net. The net $\eta$ being* ***convergent*** *is equivalent to the condition*

$$\lim_{\Xi \gg \Theta} \|\eta_\Xi - \eta_\Theta\| = 0 \qquad \forall\ \Theta, \Xi \in \mathfrak{Z}([0,T]), \Theta \subseteq \Xi. \tag{2.7}$$

*Proof.* One direction is straightforward, since every Cauchy net in $(X, \|\cdot\|)$ automatically converges. For the other direction assume $\lim_{\Xi \gg \Theta} \|\eta_\Xi - \eta_\Theta\| = 0$. It follows that for every $\varepsilon > 0$ there exists a $\Theta \in \mathfrak{Z}([0,T])$ such that $\|\eta_\Xi - \eta_\Lambda\| \leq \frac{\varepsilon}{2}$ for all $\Theta \subseteq \Xi \subseteq \Lambda$.

Hence there exists a $\Xi \in \mathfrak{Z}([0,T])$, dependent on $\Theta$ and $\varepsilon$ such that $\|\eta_\Lambda - \eta_\Theta\| \leq \frac{\varepsilon}{2}$ for $\Xi \subseteq \Lambda$. Using the sub-additivity of the norm we have for every $\Xi \subseteq \Lambda_1 \subseteq \Lambda_2$ that $\left\|\eta_{\Lambda_1} - \eta_{\Lambda_2}\right\| \leq \left\|\eta_{\Lambda_1} - \eta_\Theta\right\| + \left\|\eta_\Theta - \eta_{\Lambda_2}\right\| \leq \varepsilon$ holds and therefore obtain convergence. □

Having this new limit notation one can also simplify the convergence properties for operators between limit spaces in the sense that the norms taken are not needed to be induced from the abstract space $\mathcal{K}$ but rather from the discrete spaces $\mathcal{K}_\Theta$. This greatly simplifies our analysis, since the latter norms can be calculated explicitly in our construction.

Note that if $\mathcal{K}_\Theta$ is finite-dimensional, which will be the case in our upcoming analysis starting in chapter 3, then $\mathcal{B}(\mathcal{K}_\Theta)$ is a reflexive Banach space with the operator norm and therefore weak and weak-* topologies coincide. From now on we will assume finite-dimensionality of $\mathcal{K}_\Theta$ and therefore use the terms “weak” and “weak-*” synonymously.

**Theorem 2.17.** *Let* $V_\Theta : \mathcal{H} \longrightarrow \mathcal{K}_\Theta$ *be a net of operators on the direct system* $(\mathcal{K}_\Theta, J_{\Xi\Theta})$ *as before. The following convergence conditions for* $V_\Theta$ *are equivalent to the prior definitions. That means the net* $\Theta \longmapsto V_\Theta$ *is*

***norm convergent*** *iff* $\lim\limits_{\Xi\gg\Theta} \|V_\Xi - J_{\Xi\Theta}V_\Theta\|_{op} = 0.$

***strongly convergent*** *iff* $\lim\limits_{\Xi\gg\Theta} \|V_\Xi \,|\varphi\rangle - J_{\Xi\Theta}V_\Theta \,|\varphi\rangle\|_{\mathcal{K}_\Xi} = 0 \ \forall \ |\varphi\rangle \in \mathcal{H}.$

***weakly convergent*** *iff* $\lim\limits_{\Xi\gg\Theta} \langle\lambda_\Xi|(V_\Xi - J_{\Xi\Theta}V_\Theta)\,\varphi\rangle = 0$ *for all* $|\varphi\rangle \in \mathcal{H}$ *and all Cauchy nets* $|\lambda_\Xi\rangle \in \mathcal{K}_\Xi$.

*Let* $O_\Theta : \mathcal{K}_\Theta \longrightarrow \mathcal{K}_\Theta$ *be a net of operators. The net* $\Theta \longmapsto O_\Theta$ *is*

***norm convergent*** *iff* $\lim\limits_{\Xi\gg\Theta} \left\|O_\Xi - J_{\Xi\Theta}O_\Theta J^\dagger_{\Xi\Theta}\right\|_{op} = 0.$

***strongly convergent*** *iff* $\lim\limits_{\Xi\gg\Theta} \left\|(O_\Xi J^\dagger_\Xi - J_{\Xi\Theta}O_\Theta J^\dagger_\Theta)\,|\lambda\rangle\right\|_{\mathcal{K}_\Xi} = 0 \ \forall \ |\lambda\rangle \in \mathcal{K}.$

***weak-* convergent*** *iff* $\lim\limits_{\Xi\gg\Theta} \mathrm{tr}\Big(\rho_\Xi\left(O_\Xi - J_{\Xi\Theta}O_\Theta J^\dagger_{\Xi\Theta}\right)\Big) = 0$ *for all nets* $\rho_\Xi \in \mathfrak{T}(\mathcal{K}_\Xi)$ *which are Cauchy w.r.t. the trace norm.*

*Proof.* The proof simply uses the corollary before and some norm equivalences of the form $\|J_\Xi\,|\varphi_\Xi\rangle - J_\Theta\,|\varphi_\Theta\rangle\| = \||\varphi_\Xi\rangle - J_{\Xi\Theta}\,|\varphi_\Theta\rangle\|$. This can be shown fairly easy for any norm involved, since the norm is induced by a scalar product, the $J_{\Xi\Theta}$ are isometries and those maps constitute an inductive system. Hence one can calculate

$$\begin{aligned}\|J_\Xi\,|\varphi_\Xi\rangle - J_\Theta\,|\varphi_\Theta\rangle\|^2 &= \langle J_\Xi\varphi_\Xi - J_\Theta\varphi_\Theta|J_\Xi\varphi_\Xi - J_\Theta\varphi_\Theta\rangle \\ &= \langle J_\Xi\,(\varphi_\Xi - J_{\Xi\Theta}\varphi_\Theta)|J_\Xi\,(\varphi_\Xi - J_{\Xi\Theta}\varphi_\Theta)\rangle \\ &= \langle\varphi_\Xi - J_{\Xi\Theta}\varphi_\Theta|\varphi_\Xi - J_{\Xi\Theta}\varphi_\Theta\rangle = \||\varphi_\Xi\rangle - J_{\Xi\Theta}\,|\varphi_\Theta\rangle\|^2. \quad (2.8)\end{aligned}$$

Note that the first norm is taken w.r.t. the limit space $\mathcal{K}$, whereas the last norm is taken w.r.t. $\mathcal{K}_\Xi$, which is way less abstract. A norm on each $\mathcal{K}_\Xi$ can also be written down explicitly, due to its finite dimension.
We, split the proof into the convergence of $V_\Theta$ and $O_\Theta$ and analyze each topology separately.

**1)** $V_\Theta : \mathcal{H} \longrightarrow \mathcal{K}_\Theta$

Norm convergence:

$$V_\Theta \text{ is norm convergent} :\Longleftrightarrow \Theta \longmapsto J_\Theta V_\Theta \text{ converges on } \Big(\mathcal{B}(\mathcal{H},\mathcal{K}), \|\cdot\|_{\text{op}}\Big)$$

$$\overset{\text{Cor. 2.16}}{\Longleftrightarrow} \lim_{\Xi \gg \Theta} \|J_\Xi V_\Xi - J_\Theta V_\Theta\|_{\text{op}} = 0 \overset{\text{Eq. (2.8)}}{\Longleftrightarrow} \lim_{\Xi \gg \Theta} \|V_\Xi - J_{\Xi\Theta} V_\Theta\|_{\text{op}} = 0$$

Strong convergence:

$$V_\Theta \text{ is strongly convergent} :\Longleftrightarrow \Theta \longmapsto J_\Theta V_\Theta |\varphi\rangle \text{ converges on } (\mathcal{K}, \|\cdot\|_{\mathcal{K}}) \ \forall \ |\varphi\rangle \in \mathcal{H}$$

$$\overset{\text{Cor. 2.16}}{\Longleftrightarrow} \lim_{\Xi \gg \Theta} \|J_\Xi V_\Xi |\varphi\rangle - J_\Theta V_\Theta |\varphi\rangle\|_{\mathcal{K}} = 0 \ \forall \ |\varphi\rangle \in \mathcal{H}$$

$$\overset{\text{Eq. (2.8)}}{\Longleftrightarrow} \lim_{\Xi \gg \Theta} \|V_\Xi |\varphi\rangle - J_{\Xi\Theta} V_\Theta |\varphi\rangle\|_{\mathcal{K}_\Xi} = 0 \ \forall \ |\varphi\rangle \in \mathcal{H}$$

Weak convergence:

$$\begin{aligned} & V_\Theta \text{ is weakly convergent} \\ :\Longleftrightarrow \ & \Theta \longmapsto \langle\lambda|J_\Theta V_\Theta \varphi\rangle \text{ converges on } (\mathbb{C}, |\cdot|) \ \forall \ |\varphi\rangle \in \mathcal{H}, |\lambda\rangle \in \mathcal{K} \\ \overset{\text{Cor. 2.16}}{\Longleftrightarrow} \ & \lim_{\Xi \gg \Theta} |\langle\lambda|J_\Xi V_\Xi \varphi\rangle - \langle\lambda|J_\Theta V_\Theta \varphi\rangle| = 0 \ \forall \ |\varphi\rangle \in \mathcal{H}, |\lambda\rangle \in \mathcal{K} \end{aligned}$$

Since $|\lambda\rangle \in \mathcal{K}$ is arbitrary, we write it as $J_\Xi |\lambda_\Xi\rangle$ and require $\Xi \longmapsto |\lambda_\Xi\rangle$ to be a Cauchy net in order for $|\lambda\rangle$ to be well defined. So weak convergence is equivalent to: For all $|\varphi\rangle \in \mathcal{H}$ and all Cauchy nets $|\lambda_\Xi\rangle \in \mathcal{K}_\Xi$ we have

$$\lim_{\Xi \gg \Theta} |\langle J_\Xi \lambda_\Xi | J_\Xi (V_\Xi - J_{\Xi\Theta} V_\Theta)\varphi\rangle| = \lim_{\Xi \gg \Theta} |\langle \lambda_\Xi | (V_\Xi - J_{\Xi\Theta} V_\Theta)\varphi\rangle| = 0.$$

**2)** $O_\Theta : \mathcal{K}_\Theta \longrightarrow \mathcal{K}_\Theta$

The proof works analogously for $O_\Theta$ using the norm equation

$$\left\| J_\Xi O_\Xi J_\Xi^\dagger - J_\Theta O_\Theta J_\Theta^\dagger \right\|_{\text{op}} = \left\| J_\Xi (O_\Xi - J_{\Xi\Theta} O_\Theta J_{\Xi\Theta}^\dagger) J_\Xi^\dagger \right\|_{\text{op}} = \left\| O_\Xi - J_{\Xi\Theta} O_\Theta J_{\Xi\Theta}^\dagger \right\|_{\text{op}},$$

which reduces the abstract norm on $\mathcal{B}(\mathcal{K})$ to the norm on $\mathcal{B}(\mathcal{K}_\Xi)$. Furthermore, acting point wise on some vector it reduces the norm on $\mathcal{K}$ to the norm on $\mathcal{K}_\Xi$.

Norm convergence:

$$O_\Theta \text{ is norm convergent } :\Longleftrightarrow \Theta \longmapsto J_\Theta O_\Theta J_\Theta^\dagger \text{ converges on } \left(\mathcal{B}(\mathcal{K}), \|\cdot\|_{\text{op}}\right)$$
$$\overset{\text{Cor. 2.16}}{\Longleftrightarrow} \lim_{\Xi\gg\Theta} \left\| J_\Xi O_\Xi J_\Xi^\dagger - J_\Theta O_\Theta J_\Theta^\dagger \right\|_{\text{op}} = 0 \Longleftrightarrow \lim_{\Xi\gg\Theta} \left\| O_\Xi - J_{\Xi\Theta} O_\Theta J_{\Xi\Theta}^\dagger \right\|_{\text{op}} = 0$$

Strong convergence:

$$\begin{aligned} &O_\Theta \text{ is strong convergent} \\ &:\Longleftrightarrow \Theta \longmapsto J_\Theta O_\Theta J_\Theta^\dagger \left|\lambda\right\rangle \text{ converges on } (\mathcal{K}, \|\cdot\|_{\mathcal{K}}) \ \forall \left|\lambda\right\rangle \in \mathcal{K} \\ &\overset{\text{Cor. 2.16}}{\Longleftrightarrow} \lim_{\Xi\gg\Theta} \left\| (J_\Xi O_\Xi J_\Xi^\dagger - J_\Theta O_\Theta J_\Theta^\dagger) \left|\lambda\right\rangle \right\|_{\mathcal{K}} = 0 \ \forall \left|\lambda\right\rangle \in \mathcal{K} \\ &\Longleftrightarrow \lim_{\Xi\gg\Theta} \left\| (O_\Xi J_\Xi^\dagger - J_{\Xi\Theta} O_\Theta J_\Theta^\dagger) \left|\lambda\right\rangle \right\|_{\mathcal{K}_\Xi} = 0 \ \forall \left|\lambda\right\rangle \in \mathcal{K} \end{aligned}$$

Weak convergence:

$$\begin{aligned} &O_\Theta \text{ is weak-* convergent} \\ &:\Longleftrightarrow \Theta \longmapsto \text{tr}\left(\rho J_\Theta O_\Theta J_\Theta^\dagger\right) \text{ converges on } (\mathbb{C}, |\cdot|) \ \forall \rho \in \mathfrak{T}(\mathcal{K}) \\ &\overset{\text{Cor. 2.16}}{\Longleftrightarrow} \lim_{\Xi\gg\Theta} \left| \text{tr}\left(\rho J_\Xi O_\Xi J_\Xi^\dagger\right) - \text{tr}\left(\rho J_\Theta O_\Theta J_\Theta^\dagger\right) \right| = 0 \ \forall \rho \in \mathfrak{T}(\mathcal{K}) \end{aligned}$$

We now use the linearity and cyclicity of the trace and calculate:

$$\begin{aligned} \text{tr}\left(\rho J_\Xi O_\Xi J_\Xi^\dagger\right) - \text{tr}\left(\rho J_\Theta O_\Theta J_\Theta^\dagger\right) &= \text{tr}\left(\rho \left(J_\Xi O_\Xi J_\Xi^\dagger - J_\Xi J_{\Xi\Theta} O_\Theta J_{\Xi\Theta}^\dagger J_\Xi^\dagger\right)\right) \\ &= \text{tr}\left(J_\Xi^\dagger \rho J_\Xi \left(O_\Xi - J_{\Xi\Theta} O_\Theta J_{\Xi\Theta}^\dagger\right)\right) = \text{tr}\left(\rho_\Xi \left(O_\Xi - J_{\Xi\Theta} O_\Theta J_{\Xi\Theta}^\dagger\right)\right), \end{aligned}$$

where $\Xi \longmapsto \rho_\Xi$ must be convergent to $\rho$ in order for $\rho = J_\Xi^\dagger \rho J_\Xi$ to be a well defined object. Since the space of trace class operators $\mathfrak{T}(\mathcal{K})$ canonically carries three different norm structures, i.e. the operator norm $\|\cdot\|_{\text{op}}$, the trace norm $\|\cdot\|_{\text{tr}}$ and the Hilbert-Schmidt norm $\|\cdot\|_{\text{HS}}$, we have a priori multiple choices for the convergence of $\rho_\Xi$. But since $\mathfrak{T}(\mathcal{K})$ only with the trace norm becomes a Banach space, and therefore every Cauchy net converges, we restrict our $\rho_\Xi$ to be Cauchy w.r.t. this norm. □

We can simplify our limit notions even more, assuming the operator nets of interest being bounded. In this case we can weaken the needed conditions for convergence in the following way. For the sake of presentation we will omit the norm topology in this thesis from now on, since we won't need it in the upcoming discussion.

**Theorem 2.18.** *Let $O_\Theta : \mathcal{K}_\Theta \longrightarrow \mathcal{K}_\Theta$ be a net of operators as before and $\|O_\Theta\| \leq C$ for some $C \in \mathbb{R}$ and all $\Theta \in \mathfrak{Z}([0,T])$, then $\Theta \longmapsto O_\Theta$ is*

> ***strongly convergent*** *iff* $\lim_{\Lambda \gg \Xi} \|O_\Lambda J_{\Lambda\Theta} |\lambda_\Theta\rangle - J_{\Lambda\Xi} O_\Xi J_{\Xi\Theta} |\lambda_\Theta\rangle\|_{\mathcal{K}_\Lambda} = 0$ *for all Cauchy nets* $|\lambda_\Theta\rangle \in \mathcal{K}_\Theta$.
>
> ***weakly convergent*** *iff* $\lim_{\Lambda \gg \Xi} \left\langle \lambda_\Theta \middle| \left( J^\dagger_{\Lambda\Theta} O_\Lambda J_{\Lambda\Theta} - J^\dagger_{\Xi\Theta} O_\Xi J_{\Xi\Theta} \right) \lambda_\Theta \right\rangle = 0$ *for all* $|\lambda_\Theta\rangle \in \mathcal{K}_\Theta$.

*Proof.* The basic proof idea is to exploit the fact that we can approximate every vector $\lambda \in \mathcal{K}$ by a net $\lambda_\Theta$ and use the triangle inequality.

**Strong convergence** is equivalent to $\lim_{\Lambda \gg \Xi} \left\| (J_\Lambda O_\Lambda J^\dagger_\Lambda - J_\Xi O_\Xi J^\dagger_\Xi) |\lambda\rangle \right\|_{\mathcal{K}} = 0$ for every $|\lambda\rangle \in \mathcal{K}$. Since every Cauchy net in $\mathcal{K}$ converges and $|\lambda\rangle \in \mathcal{K}$ above is arbitrary, we can show strong convergence on elements of the form $J_\Theta |\lambda_\Theta\rangle$ for Cauchy nets $\Theta \longmapsto |\lambda_\Theta\rangle$ and assume that $|\lambda_\Theta\rangle$ converges to $|\lambda\rangle$ w.r.t. $\|\cdot\|_{\mathcal{K}}$. We then calculate:

$$
\begin{aligned}
&\left\| \left( J_\Lambda O_\Lambda J^\dagger_\Lambda - J_\Xi O_\Xi J^\dagger_\Xi \right) |\lambda\rangle \right\|_{\mathcal{K}} \\
&\quad = \left\| \left( J_\Lambda O_\Lambda J^\dagger_\Lambda - J_\Xi O_\Xi J^\dagger_\Xi \right) \left( |\lambda\rangle - J_\Theta |\lambda_\Theta\rangle + J_\Theta |\lambda_\Theta\rangle \right) \right\|_{\mathcal{K}} \\
&\quad \leq \left\| J_\Lambda O_\Lambda J^\dagger_\Lambda \left( |\lambda\rangle - J_\Theta |\lambda_\Theta\rangle \right) \right\|_{\mathcal{K}} + \left\| J_\Xi O_\Xi J^\dagger_\Xi \left( |\lambda\rangle - J_\Theta |\lambda_\Theta\rangle \right) \right\|_{\mathcal{K}} \\
&\qquad\qquad + \left\| J_\Lambda O_\Lambda J^\dagger_\Lambda J_\Theta |\lambda_\Theta\rangle - J_\Xi O_\Xi J^\dagger_\Xi J_\Theta |\lambda_\Theta\rangle \right\|_{\mathcal{K}} \\
&\quad \leq 2C \underbrace{\| |\lambda\rangle - J_\Theta |\lambda_\Theta\rangle \|_{\mathcal{K}}}_{\longrightarrow 0} + \| J_\Lambda O_\Lambda J_{\Lambda\Theta} |\lambda_\Theta\rangle - J_\Lambda J_{\Lambda\Xi} O_\Xi J_{\Xi\Theta} |\lambda_\Theta\rangle \|_{\mathcal{K}} \\
&\quad \leq \| O_\Lambda J_{\Lambda\Theta} |\lambda_\Theta\rangle - J_{\Lambda\Xi} O_\Xi J_{\Xi\Theta} |\lambda_\Theta\rangle \|_{\mathcal{K}_\Lambda}
\end{aligned}
$$

Therefore the strong convergence of a net of bounded operators $O_\Theta$ is equivalent to the condition $\lim_{\Lambda \gg \Xi} \|O_\Lambda J_{\Lambda\Theta} |\lambda_\Theta\rangle - J_{\Lambda\Xi} O_\Xi J_{\Xi\Theta} |\lambda_\Theta\rangle\|_{\mathcal{K}_\Lambda} = 0$ for every Cauchy net $|\lambda_\Theta\rangle \in \mathcal{K}_\Theta$.

**Weak convergence** is given by $\lim_{\Lambda \gg \Xi} \mathrm{tr}\left(\rho \left( J_\Lambda O_\Lambda J^\dagger_\Lambda - J_\Xi O_\Xi J^\dagger_\Xi \right)\right) = 0$. Here we can restrict the state $\rho$ to pure states, since the set of states with finite rank is dense in $\mathfrak{T}(\mathcal{K})$ w.r.t. the trace norm and every trace class operator is a convex combination of pure states.

Hence for every $\rho \in \mathfrak{T}(\mathcal{K})$ and $\varepsilon > 0$ we can find a natural number $n \in \mathbb{N}$ and a family of vectors $\{|\lambda_i\rangle \in \mathcal{K}_\Theta \mid 1 \leq i \leq n\}$, s.t. the operator

$$\rho_\Theta := \sum_{i=1}^{n} |\lambda_i\rangle\langle\lambda_i| \in \mathfrak{T}(\mathcal{K}_\Theta)$$

is $\varepsilon$ close to $\rho$, i.e. $\|\rho - J_\Theta \rho_\Theta\|_{\mathrm{tr}} = \mathrm{tr}(|\rho - J_\Theta\rho_\Theta|) \leq \varepsilon$. Similar to before and using the cyclicity of the trace we compute

$$\begin{aligned}
\mathrm{tr}\Big(\rho\left(J_\Lambda O_\Lambda J_\Lambda^\dagger - J_\Xi O_\Xi J_\Xi^\dagger\right)\Big) &= \mathrm{tr}\Big((\rho - J_\Theta\rho_\Theta + J_\Theta\rho_\Theta)\left(J_\Lambda O_\Lambda J_\Lambda^\dagger - J_\Xi O_\Xi J_\Xi^\dagger\right)\Big) \\
&= \mathrm{tr}\Big((\rho - J_\Theta\rho_\Theta)\left(J_\Lambda O_\Lambda J_\Lambda^\dagger - J_\Xi O_\Xi J_\Xi^\dagger\right)\Big) + \mathrm{tr}\Big(J_\Theta\rho_\Theta\left(J_\Lambda O_\Lambda J_\Lambda^\dagger - J_\Xi O_\Xi J_\Xi^\dagger\right)\Big) \\
&\leq 2C\underbrace{\mathrm{tr}(\rho - J_\Theta\rho_\Theta)}_{\longrightarrow 0} + \mathrm{tr}\Big(\rho_\Theta\left(J_\Lambda O_\Lambda J_\Lambda^\dagger - J_\Xi O_\Xi J_\Xi^\dagger\right)J_\Theta\Big) \\
&= \mathrm{tr}\Big(\rho_\Theta\left(J_\Lambda O_\Lambda J_{\Lambda\Theta}^\dagger - J_\Xi O_\Xi J_{\Xi\Theta}^\dagger\right)\Big) = \sum_{i=1}^{n}\Big\langle\lambda_i\Big|\left(J_\Lambda O_\Lambda J_{\Lambda\Theta}^\dagger - J_\Xi O_\Xi J_{\Xi\Theta}^\dagger\right)\lambda_i\Big\rangle .
\end{aligned}$$

Hence $\lim_{\Lambda\gg\Xi} \mathrm{tr}\Big(\rho\left(J_\Lambda O_\Lambda J_\Lambda^\dagger - J_\Xi O_\Xi J_\Xi^\dagger\right)\Big) = 0$ for all $\rho \in \mathfrak{T}(\mathcal{K})$ is equivalent to the condition $\lim_{\Lambda\gg\Xi}\Big\langle\lambda_\Theta\Big|\left(J_\Lambda O_\Lambda J_{\Lambda\Theta}^\dagger - J_\Xi O_\Xi J_{\Xi\Theta}^\dagger\right)\lambda_\Theta\Big\rangle = 0$ for every $|\lambda_\Theta\rangle \in \mathcal{K}_\Theta$, iff $O_\Theta$ is bounded. □

However, in some proofs we will stumble upon, those proof-techniques aren't very effective. In those settings it will be more convenient to restrict our proofs to a suitable dense subspace of nets. Firstly we will define this new kind of strong and weak convergence. Secondly we show equivalence to the already defined notions of convergence for our kind of underlying algebraic structures.

Since the limit space of the inductive system is constructed to be the set of Cauchy nets modulo Null nets it is clear that for every Cauchy net $|\varphi_\Theta\rangle$ it follows that $J_\Theta|\varphi_\Theta\rangle \in \mathcal{K}$ is convergent, as we have seen before. Exploiting this fact about the limit space motivates the following definition:

**Definition 2.19.** *Let $A_\Theta \in \mathcal{B}(\mathcal{K}_\Theta)$ be a net of bounded operators for $\Theta \in \mathfrak{Z}([0,T])$. We say that $A_\Theta$ is:*

$\widetilde{\textbf{\textit{strong}}}$ ***convergent*** *iff $\forall\,|\varphi_\Theta\rangle \in \mathcal{K}_\Theta$ Cauchy $\Rightarrow A_\Theta|\varphi_\Theta\rangle \in \mathcal{K}_\Theta$ Cauchy.*

$\widetilde{\textbf{\textit{weak}}}$ ***convergent*** *iff $\forall\,|\lambda_\Theta\rangle, |\psi_\Theta\rangle \in \mathcal{K}_\Theta$ Cauchy $\Rightarrow \langle\lambda_\Theta|A_\Theta\psi_\Theta\rangle \in \mathbb{C}$ Cauchy.*

On the algebraic structures we are going to look at, this new version of strong and weak convergence are equivalent to the prior ones.

**Theorem 2.20.** *Let $(\mathcal{K}_\Theta, J_{\Xi\Theta})$ be an inductive system of Hilbert spaces. Then the following equivalences hold: Let $\Theta \longmapsto A_\Theta \in \mathcal{B}(\mathcal{K}_\Theta)$ be a net of bounded operators, then we have*

$$A_\Theta \textit{ is strong convergent} \iff A_\Theta \textit{ is } \widetilde{\textit{strong}} \textit{ convergent}$$
$$A_\Theta \textit{ is weak convergent} \iff A_\Theta \textit{ is } \widetilde{\textit{weak}} \textit{ convergent}$$

*Proof.* We show the two equivalences separately.

Strong convergence:

$A_\Theta$ is $\widetilde{\text{strong}}$ convergent by definition iff the net $\Theta \longmapsto |\varphi_\Theta\rangle \in \mathcal{K}_\Theta$ being Cauchy implicates that

$$\forall \varepsilon > 0 \; \exists \, \Theta, \Xi \in \mathfrak{Z}([0,T]) \text{ s.t. } \| J_\Xi A_\Xi |\varphi_\Xi\rangle - J_\Theta A_\Theta |\varphi_\Theta\rangle \|_{\mathcal{K}} \leq \varepsilon,$$

i.e. the net $\Theta \longmapsto A_\Theta |\varphi_\Theta\rangle$ being Cauchy. Rearranging this norm above, we see

$$\| J_\Xi A_\Xi |\varphi_\Xi\rangle - J_\Theta A_\Theta |\varphi_\Theta\rangle \|_{\mathcal{K}} = \| A_\Xi J_{\Xi\Theta} |\varphi_\Theta\rangle - J_{\Xi\Theta} A_\Theta |\varphi_\Theta\rangle \|_{\mathcal{K}_\Xi}$$

with a well defined object $J_{\Xi\Theta} |\varphi_\Theta\rangle$ since $|\varphi_\Theta\rangle$ is assumed to be Cauchy. Therefore

$$\forall \varepsilon > 0 \; \exists \, \Theta, \Xi \in \mathfrak{Z}([0,T]) \text{ s.t. } \| A_\Xi J_{\Xi\Theta} |\varphi_\Theta\rangle - J_{\Xi\Theta} A_\Theta |\varphi_\Theta\rangle \|_{\mathcal{K}_\Xi} \leq \varepsilon,$$

but because of the completeness of $\mathcal{K}_\Xi$ this net converges too. Furthermore by corollary 2.16 this is equivalent to

$$\lim_{\Xi \gg \Theta} \| A_\Xi J_{\Xi\Theta} |\varphi_\Theta\rangle - J_{\Xi\Theta} A_\Theta |\varphi_\Theta\rangle \|_{\mathcal{K}_\Xi} = 0,$$

which is the strong convergence condition for bounded $A_\Theta$, as seen in theorem 2.18.

Weak convergence:

The proof for weak convergence works completely analogous, using the equation

$$|\langle\varphi_\Lambda|A_\Lambda\psi_\Lambda\rangle - \langle\varphi_\Xi|A_\Xi\psi_\Xi\rangle| = \left|\left\langle\varphi_\Theta\middle|\left(J^\dagger_{\Lambda\Theta}A_\Lambda J_{\Lambda\Theta} - J^\dagger_{\Xi\Theta}A_\Xi J_{\Xi\Theta}\right)\psi_\Theta\right\rangle\right|.$$

This term being Cauchy is then equivalent to the $\lim\limits_{\Lambda\gg\Xi}$ being zero for all Cauchy nets $|\varphi_\Theta\rangle, |\psi_\Theta\rangle \in \mathcal{K}_\Theta$. □

## 2.4 $L^p$-Spaces and Approximation

In this thesis multiple versions of integrals will arise throughout our limit constructions. We won't use distinct notation, rather than denoting every integral in the same way. However, if it is necessary or crucial for our discussion, we will say whether the integral is meant to be Riemann, Lebesgue or Bochner.

Since Bochner integrals are not part of the standard literature, we will define them in the following way. Lebesgue integrable functions $L^p(S,\Omega,\mu,\mathbb{K}) =: L^p(S,\mathbb{K})$ are defined for functions $f : (S,\Omega,\mu) \longrightarrow \mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ on a measurable space $(S,\Omega,\mu)$. It seems possible to lift this to functions with a more general image space.

**Definition 2.21** (Bochner spaces)**.** *We define the spaces of* ***p-times Bochner integrable functions****, for Banach space valued functions $f : (S,\Omega,\mu) \longrightarrow (X, \|\cdot\|_X)$ via*

$$L^p(S,\Omega,\mu,X) =: L^p(S,X) = \left\{f : (S,\Omega,\mu) \longrightarrow (X,\|\cdot\|_X) \;\middle|\; \|f\|_{L^p(S,X)} < \infty\right\}$$
$$\text{with} \quad \|f\|_{L^p(S,X)} = \left(\int_S \|f(t)\|_X^p \,\mathrm{d}\mu(t)\right)^{1/p} \qquad \text{if} \quad 1 \le p < \infty$$
$$\text{and} \quad \|f\|_{L^\infty(S,X)} = \operatorname*{ess\,sup}_{t\in S} \|f(t)\|_X.$$

*As usual one implicitly identifies functions modulo $\mu$-zero sets. Almost all notions of usual $L^p$ spaces extend to the Bochner $L^p$ spaces, especially the fact that $L^2(S,X)$ is a Hilbert space if $X$ is a Hilbert space.*

Having this it makes sense for us to define, for example, an object like $U \in L^p([0,T],(\mathcal{B}(\mathcal{H}),\|\cdot\|_{\mathrm{op}}))$. Another important property about $L^p$ functions are their subsets. Let us analyze this a bit more carefully. As seen in [RF88][Section 7.4., Proposition 10] we have the following statement.

**Proposition 2.22.** *Let $[a, b] \subset \mathbb{R}$ be a closed and bounded interval and $1 \leq p < \infty$. Then the subspace of step functions w.r.t. the (countable) index set $I$, denoted as*

$$Step_I([0,T],\mathbb{R}) := \left\{ f : [0,T] \longrightarrow \mathbb{R} \;\middle|\; f = \sum_{i \in I} \alpha_i \, \chi_{[t_i, t_{i+1})}, \;\; \alpha_i \in \mathbb{R} \right\}, \tag{2.9}$$

*is dense in $L^p([a,b],\mathbb{R})$ w.r.t. to the p-norm.*

This denseness is, by far, the most important for this thesis. A lot of theorems involving $L^p$ functions will be proven on step functions first and then "lifted" using sequences of step functions converging to general $L^p$ elements. If those theorems are proven on $L^p$, then they automatically hold on test function spaces as well, since they are subsets of $L^p$ for $1 \leq p < \infty$.
By tensoring, we can also lift the image space from $\mathbb{R}$ to be some Hilbert space $\mathcal{K}$. The most important dense subspaces are summarized in the following figure:

For $1 \leq p < \infty$

$$\begin{array}{ccc} \mathrm{Step}_I([0,T],\mathcal{K}) & & \\ \cap & & \\ (L^p([0,T],\mathcal{K}), \|\cdot\|_p) & \supset & \mathscr{S}([0,T],\mathcal{K}) \\ \cup & & \cup \\ C([0,T],\mathcal{K}) & & C_c^\infty([0,T],\mathcal{K}) \end{array}$$

Figure 2.2: Dense subspaces of $L^p([0,T],\mathcal{K})$

Here we wrote for the space of continuous functions $C([0,T],\mathcal{K})$ and denoted the test function spaces of smooth, compactly supported functions as $C_c^\infty([0,T],\mathcal{K})$ and Schwartz functions as $\mathscr{S}([0,T],\mathcal{K})$.

Some theorems that we will stumble upon, will canonically produce Riemann integrals, we therefore need to mention the following detail: It is known that Riemann integrals are defined via converging sequences of upper and lower sums. One can, however, generalize this idea from sequences to nets. In the case, where we consider integral decompositions as the directed set behind our nets, i.e. $(\mathfrak{Z}([0,T]), \subseteq)$, the basic motivation behind the Riemann integral seems preserved. Indeed, it is a fruitful idea to define for $\alpha : [0,T] \longrightarrow \mathbb{C}$

$$\int_0^T \alpha(t)\,\mathrm{d}t := \varinjlim_{\Theta \in \mathfrak{Z}([0,T])} \sum_{i \in I(\Theta)} \tau_i \alpha(t_i), \tag{2.10}$$

if the "upper and lower sums" on the right hand side converge. It is quite easy to see, that this convergence is guaranteed using, for example, piece wise continuous functions $\alpha(t)$.

## 2.5 Evolution Systems and Master Equations

### Master Equations and Lindbladians

**Master equations** arise naturally in many body physics and applied science like chemistry and biology. They provide a method to understand the dynamics of a system with many possible states and transitions between them.

In contrast, a **quantum master equation** is a more general notion, because normal (or classical) master equations are first order differential equations of transition probabilities. In a quantum theory, those would correspond to the diagonal elements of a density matrix, but a quantum master equation is a set of differential equations of the full density matrix and therefore also describes decoherence and entanglement.

An important example for a quantum master equation, describing so-called **Markovian** systems, is the **Lindblad equation**. A system is called Markovian, if it fulfills the **Markov property**, i.e. given a state at time $t$, then the evolution is completely determined by this time and does not depend on any time $t' \leq t$. The system is in a certain sense "memoryless". It was shown in [Dav74] that all master equations describing a system, being weakly coupled to its surrounding, must be Markovian.

But before we analyze Lindblad equations in greater detail in the next chapter we need to define some other preliminaries.

## Semigroups and Evolution Systems

Semigroups and their generators have been studied a lot in the last years and their application to quantum mechanics, and theoretical physics in general, yielded extraordinary results. Especially when trying to formalize time evolution, those notions arise quite naturally as we will see now.

**Definition 2.23.** *Let* $(X, \|\cdot\|_X)$ *be a Banach space. A map* $U : \mathbb{R}_+ := [0, \infty) \longrightarrow \mathcal{B}(X)$ *with the properties*

*i)* $U(0) = \mathbb{1}_X$,

*ii)* $U(t+s) = U(t) \circ U(s) \quad \forall s, t \in \mathbb{R}_+$,

*is called a* ***one-parameter semigroup****, or just* ***semigroup****. Since the spaces* $\mathcal{B}(X)$ *and* $X$ *canonically carry topologies and additional algebraic structures as well, one can define some more concepts on a semigroup by defining that* $U$ *is*

***uniformly continuous*** *if* $\lim_{t \searrow 0} \|U(t) - \mathbb{1}_X\|_{op} = 0$.

***strongly continuous*** *if* $\lim_{t \searrow 0} \|U(t)x - x\|_X = 0 \quad \forall x \in X$ .

***contractive*** *if* $\|U(t)\|_{op} \leq 1 \quad \text{for} \quad 0 \leq t \in \mathbb{R}_+$ .

***unital*** *if* $U(t)(\mathbb{1}_X) = \mathbb{1}_X \quad \forall t \in \mathbb{R}_+$ .

*Whenever the limit*

$$Gx = \lim_{t \searrow 0} \frac{1}{t} \left(U(t) - \mathbb{1}_X\right) x \tag{2.11}$$

*exists we call* $G$ *the* ***infinitesimal generator*** *of the semigroup. In the uniformly continuous case the generator exists on all of* $X$ *and one can recover the whole group via its generator by defining*

$$U(t) = e^{Gt} := \sum_{n=0}^{\infty} \frac{G^n}{n!} t^n. \tag{2.12}$$

Evolution systems[3] are in a certain sense just the two parameter version of semigroups, as one can see fairly easy:

**Definition 2.24.** *Let $(X, \|\cdot\|_X)$ be a Banach space and $0 \leq r \leq s \leq t \in \mathbb{R}$. A two-parameter family of maps $\mathbb{E}(s,t) : X \longrightarrow X$ is called an* ***evolution system*** *iff:*

$$\left.\begin{aligned} \mathbb{E}(r,s) \circ \mathbb{E}(s,t) = & \quad \mathbb{E}(r,t) \\ \mathbb{E}(t,t) = & \quad \mathbb{1}_X \end{aligned}\right\} \tag{2.13}$$

*An evolution system is said to be*

***norm continuous:*** *iff $(s,t) \longmapsto \mathbb{E}(s,t)$ is almost-everywhere continuous w.r.t. the operator norm on $\mathcal{B}(X)$.*

***strong continuous:*** *iff $(s,t) \longmapsto \mathbb{E}(s,t)(B)$ is almost-everywhere continuous for all $B \in X$ w.r.t. the norm topology on $X$.*

***contractive:*** *iff $\|\mathbb{E}(s,t)\| \leq 1 \quad \forall\, 0 \leq s \leq t$.*

***unital:*** *iff $\mathbb{E}(s,t)(\mathbb{1}_X) = \mathbb{1}_X$.*

*And if, additionally, $X$ has a predual space, denoted by $X_*$, then $\mathbb{E}(s,t)$ is called*

***weak-* continuous:*** *iff $(s,t) \longmapsto \rho \circ \mathbb{E}(s,t)(B)$ is almost-everywhere continuous for all $\rho \in X_*$ and $B \in X$.*

All those notions can be extended to unbounded evolutions,[4] restricting the domain to a (dense) subspace $\mathcal{D} \subset X$.
The continuity of the semigroup/evolution system is a very crucial information, mostly seen in the existence and form of generators, which we will analyze in the upcoming section.

[3] Physicists also like to call evolution systems "propagators", especially in the background of time evolution in QFT.

[4] Except norm convergence, which is clearly not defined in this case.

## Norm Continuous Semigroups

Probably the most important theorem concerning norm continuous semigroups is the following.

**Theorem 2.25** (Lindblad theorem)**.** *Let $\mathcal{H}$ be a separable Hilbert space and $\mathbb{E}(t) : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ for $t \geq 0$ be a norm continuous semigroup of completely-positive and unital maps. Then one can find bounded operators $K : \mathcal{H} \longrightarrow \mathcal{H}$ and $L : \mathcal{H} \longrightarrow \mathcal{K} \otimes \mathcal{H}$, where $\mathcal{K}$ is a separable Hilbert space, such that the generator $\mathcal{L}$ of $\mathbb{E}(t)$ is given by:*

$$\left.\begin{aligned} \mathcal{L}(B) &= K^{\dagger}B + BK + L^{\dagger}\left(\mathbb{1}_{\mathcal{K}} \otimes B\right)L \\ \mathcal{L}\left(\mathbb{1}_{\mathcal{H}}\right) &= 0 \end{aligned}\right\} \tag{2.14}$$

*for $B \in dom(\mathcal{L}) = \mathcal{B}(\mathcal{H})$. On the contrary, every operator satisfying these two equations generates a completely-positive unital and norm continuous semigroup.*

The key significance about this theorem is that it reduces the complexity of finding quantum channels from characterizing operators on the whole observable algebra $\mathcal{B}(\mathcal{H})$ via a few bounded operators on $\mathcal{H}$.

## Strongly Continuous Semigroups

In open quantum systems strongly continuous semigroups of completely positive maps are of key importance and often emerge as the natural description of most physically interesting systems. The study of those semigroups is therefore a field of active research for decades. Surprisingly, there is no direct analogue for the Lindblad theorem for this case, but some strong partial results, especially regarding gauge symmetries, exist. The following theorem would be the easiest and most comprehensive statement, about unbounded (and therefore not norm continuous) Lindbladians.

**Theorem 2.26.** *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, $\mathcal{K}$ being separable. Furthermore let $K : \mathcal{H} \supset dom(K) \longrightarrow \mathcal{H}$ be the generator of a one parameter contraction semigroup and $L : dom(K) \subset dom(L) \longrightarrow \mathcal{K} \otimes \mathcal{H}$ being an operator satisfying the* ***infinitesimal conservativity condition***

$$\left\|L\psi\right\|^2 \leq -2\,\mathrm{Re}\,\langle\psi|K\psi\rangle \quad \forall\ |\psi\rangle \in dom(K). \tag{2.15}$$

*Then there exists a unique weak-* continuous contraction semigroup $\mathbb{E}_{min}(t)$ on $\mathcal{B}(\mathcal{H})$ called the **minimal solution**, solving the Cauchy equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \psi | \mathbb{E}_{min}(t)(B)\psi \right\rangle = \left\langle \psi | \mathcal{L}(\mathbb{E}_{min}(t)(B))\psi \right\rangle , \tag{2.16}$$

*where the generator $\mathcal{L}$ is determined by*

$$\left\langle \psi | \mathcal{L}(B)\psi \right\rangle = \left\langle K\psi | B\psi \right\rangle + \left\langle \psi | BK\psi \right\rangle + \left\langle L\psi | \left(\mathbb{1}_{\mathcal{K}} \otimes B\right) L\psi \right\rangle . \tag{2.17}$$

*Proof.* One proof is outlined by Alexander Holevo in [Hol95b] and should not be subject of this thesis. □

One can easily see that the resulting Lindbladians get way more complicated having only strong convergence. Learning more about the behavior of those evolution systems is still a field of active research.

As in the case of semigroups, where one can define a generator, one can define two different generators corresponding to an evolution system. Those, usually unbounded, operators are in most cases related via some initial conditions, but can, however, be arbitrarily wild after all.

## Gauge Symmetry of Lindblad Equations

The main idea of exploiting the gauge symmetry of the Lindblad equation and its relation to semigroups was first formalized by Alexander Holevo in [Hol95a]. It is obvious that, given a Lindblad generator $\mathcal{L}$, the choice of operators $L$ and $K$ is far from unique. More precisely:

**Definition 2.27.** *Let $K(t) : \mathcal{H} \supset dom(K) \longrightarrow \mathcal{H}$ and $L(t) : dom(L) \longrightarrow \mathcal{K} \otimes \mathcal{H}$ define a Lindbladian as in equation (2.17).*

*For every $t \in [0,T]$ let $x(t) \in \mathbb{R}$, $\left|\lambda(t)\right\rangle \in \mathcal{K}$ and $U(t) \in \mathcal{B}(\mathcal{H})$ be a unitary operator valued function. We say the following family of operators constitutes a **gauge shifted Lindbladian** with **gauge triple** $(U(t), \left|\lambda(t)\right\rangle, x(t))$:*

$$\widetilde{L}(t) = \left(U(t) \otimes \mathbb{1}_{\mathcal{H}}\right) L(t) + \left|\lambda(t)\right\rangle \otimes \mathbb{1}_{\mathcal{H}} \tag{2.18}$$

$$\widetilde{K}(t) = K(t) - \frac{1}{2}\|\lambda(t)\|^2 - \left(\left\langle U(t)^{\dagger}\lambda(t)\right| \otimes \mathbb{1}_{\mathcal{H}}\right) L(t) + ix(t)\mathbb{1}_{\mathcal{H}} \tag{2.19}$$

Explicitly calculating $\widetilde{\mathcal{L}}(t)$ shows that it is identical to $\mathcal{L}(t)$ when acting on arbitrary vectors, i.e. $\langle\varphi|\mathcal{L}(B)\psi\rangle = \left\langle\varphi\middle|\widetilde{\mathcal{L}}(B)\psi\right\rangle$ for all $|\psi\rangle, |\varphi\rangle \in \mathrm{dom}(K)$ (see [Neu15][Lemma 3.11., p. 53].).

It is way harder to proof that $\mathrm{dom}(K) = \mathrm{dom}(\widetilde{K})$ and that $\widetilde{\mathcal{L}}(t)$ is actually a standard Lindblad generator. One also needs to show whether the associated Cauchy equation is solvable or not, or whether the function $t \longmapsto \widetilde{\mathcal{L}}(t)$ is still continuous.

For arbitrary gauge triples this would not be the case. We need to add more structure in the following form to fulfill all the mentioned requirements.

**Definition 2.28.** *Let $(U(t), |\lambda(t)\rangle, x(t))$ be a gauge triple. We say the gauge triple is **continuously differentiable** iff the function:*

$$\begin{aligned}
|\lambda(t)\rangle : [0,T] &\longrightarrow \mathcal{K}, & t &\mapsto \lambda(t) & &\textit{is cont. differentiable}\\
U(t) : [0,T] &\longrightarrow \mathcal{B}(\mathcal{K}), & t &\mapsto U(t) & &\textit{is cont. differentiable in the strong topology}\\
x(t) : [0,T] &\longrightarrow \mathbb{R}, & t &\mapsto x(t) & &\textit{is cont. differentiable in norm}
\end{aligned}$$

Having this it can be shown that:

**Lemma 2.29.** *Let $(U(t), |\lambda(t)\rangle, x(t))$ be a continuously differentiable gauge and assume that $\mathrm{dom}(K(t)) = \mathrm{dom}(K(0))$ for all $t \in [0,T]$ and set $\mathcal{D} = \mathrm{dom}(K(0))$. The family of gauged operators $\widetilde{K}(t)$ generates an $\mathcal{D}$-valued evolution system and the Cauchy equation with Lindbladian $\widetilde{\mathcal{L}}$ possesses a minimal solution.*

*Proof.* The proof can be found in [Neu15][Lemma 3.13., p. 53]. □

In this thesis the evolution of an open quantum system will be given by a quantum channel, i.e. a positive and unital (resp. trace preserving) map between operator algebras, i.e. the algebra of states (resp. the algebra of observables). The two different notions come from the distinct ways of describing an evolution of a quantum system, that is Heisenberg, or Schrödinger picture. In this thesis the Heisenberg picture will be used more often, so a quantum channel maps from the bounded operators of a Hilbert space $\mathcal{B}(\mathcal{H})$ into itself.

## 2.6 The Stinespring Factorization Theorem

The **Stinespring factorization theorem** is an important result of operator theory, proven by W. Forrest Stinespring in 1955 (see [Sti55]). It shows that every completely positive map between $C^*$-algebras $\mathcal{A}, \mathcal{B}$ can be written as a concatenation of two maps of the form:

i) A $^*$-representation $\pi$ of the C$^*$-algebra $\mathcal{A}$ using some auxiliary Hilbert space $\mathcal{K}$, i.e. a $^*$-homomorphism

$$\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{K}).$$

ii) An operator map of the form $X \longmapsto V^\dagger X V$ with a map $V : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{K})$.

This theorem is extremely important, since it fully characterizes the form of every completely positive map between C$^*$-algebras, i.e. quantum channels, via some bounded operators and an auxiliary Hilbert space. The relevance of this theorem is in this sense comparable to the Lindblad theorem. Introducing this more formally, we have:

**Theorem 2.30** (Stinespring Factorization Theorem)**.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $\widehat{\mathbb{E}}, \widehat{\mathbb{F}} : \mathcal{B}(\mathcal{H}_2) \longrightarrow \mathcal{B}(\mathcal{H}_1)$ be completely positive normal unital maps. Then there exists a separable Hilbert space $\mathcal{K}$, called the **dilation space**, with identity operator $\mathbb{1}_\mathcal{K}$ and an operator of the form $V : \mathcal{H}_1 \longrightarrow \mathcal{K} \otimes \mathcal{H}_2$ such that:*

*i) For all $B \in \mathcal{B}(\mathcal{H}_2)$ we have $\widehat{\mathbb{E}}(B) = V^\dagger(\mathbb{1}_\mathcal{K} \otimes B)V$.*

*ii) $V$ and $\mathcal{K}$ can be chosen **minimal**. In this case:*

$$\mathcal{K} \otimes \mathcal{H}_2 = \overline{span}\left\{(\mathbb{1}_\mathcal{K} \otimes X)V\left|\varphi\right\rangle \;\middle|\; \left|\varphi\right\rangle \in \mathcal{H}_1, X \in \mathcal{B}(\mathcal{H}_2)\right\}, \tag{2.20}$$

*and for any non minimal dilation $\widetilde{V} : \mathcal{H}_1 \longrightarrow \widetilde{\mathcal{K}} \otimes \mathcal{H}_2$ there exists an unique isometry $W : \mathcal{K} \longrightarrow \widetilde{\mathcal{K}}$, such that $\widetilde{V}\left|\varphi\right\rangle = (W \otimes \mathbb{1}_{\mathcal{H}_2})V\left|\varphi\right\rangle$ for all $\left|\varphi\right\rangle \in \mathcal{H}_1$.*

*iii) If $\widehat{\mathbb{E}} - \widehat{\mathbb{F}}$ is completely positive and the Stinespring dilation of $\mathbb{E}$ is given by*

$$\widehat{\mathbb{E}}(B) = V^\dagger(\mathbb{1}_\mathcal{K} \otimes B)V \tag{2.21}$$

*then there exists an unique positive operator $F \in \mathcal{B}(\mathcal{K})$ with $0 \leq F \leq \mathbb{1}$, such that $\widehat{\mathbb{F}}(B) = V^\dagger(F \otimes B)V$. This property is called the **Radon-Nikodym property** of the Stinespring dilation.*

A proof for the Radon-Nikodym property can by found in [Rag03][Corollary 3.2]. There is an important alternative in the formulation of the Stinespring theorem using so-called **Kraus operators**. If one fixes an (at most countable) orthonormal basis of $\mathcal{K}$, i.e. $\{|\alpha\rangle \in \mathcal{K} \,|\, \alpha \in I\}$ with $\mathbb{1} = \sum_{\alpha \in I} |\alpha\rangle\langle\alpha|$, [EK98] has shown the following identity:

**Corollary 2.31.** *Under the assumptions of theorem 2.30 one can find a family of operators $V_\alpha : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ for countable $\alpha \in I$ s.t.*

$$\widehat{\mathbb{E}}(X) = \sum_{\alpha \in I} V_\alpha^\dagger X V_\alpha \tag{2.22}$$

*and the sum converges in weak-$*$ topology.*

The connection between both descriptions can be seen easily, since by fixing a basis $|\alpha\rangle$ of $\mathcal{K}$ one can construct a Stinespring isometry $V$ via the Kraus operators $V_\alpha$ as follows

$$V : \mathcal{H}_1 \longrightarrow \mathcal{K} \otimes \mathcal{H}_2, \qquad |\varphi\rangle \longmapsto \sum_{\alpha \in I} |\alpha\rangle \otimes V_\alpha |\varphi\rangle\,. \tag{2.23}$$

The Stinespring factorization theorem had even wider consequences, regarding the modern formalism of measurements in quantum information theory.

## 2.7 Measurements in Quantum Mechanics

The Radon-Nikodym relation is remarkable, since this property gives rise to a certain one-to-one correspondence between quantum channels and measurements. More formally:

Given a resolution of the identity on $\mathcal{K}$ there exists a decomposition of any quantum channel $\widehat{\mathbb{E}} \in \mathfrak{CP}(\mathcal{B}(\mathcal{H}_2), \mathcal{B}(\mathcal{H}_1))$ with Stinespring dilation $\widehat{\mathbb{E}}(X) = V^\dagger(\mathbb{1}_\mathcal{K} \otimes X)V$ into the following form using Kraus operators

$$\widehat{\mathbb{E}}(X) = \sum_{\alpha \in I} \widehat{\mathbb{E}}_\alpha(X) := \sum_{\alpha \in I} V_\alpha^\dagger X V_\alpha. \tag{2.24}$$

We can find (see [Rag03][Theorem 3.3]) unique positive operators $E_\alpha \in \mathcal{B}(\mathcal{K})$ with $\sum\limits_{\alpha \in I} E_\alpha = \mathbb{1}_{\mathcal{B}(\mathcal{K})}$ s.t.

$$\widehat{\mathbb{E}}_\alpha(X) = V^\dagger(E_\alpha \otimes X)V. \tag{2.25}$$

This decomposition motivates the definition of a POVM.

**Definition 2.32.** *A **positive operator valued (probability) measure**, or short **POVM**, over the measure space $(\Omega, \mathcal{F})$ and Hilbert space $\mathcal{H}$ is a map $M : \mathcal{F} \longrightarrow \mathcal{B}(\mathcal{H})$ satisfying the following conditions*

*i)* *$M(F) > 0$ for all $F \in \mathcal{F}$ and $M(\emptyset) = 0$,*

*ii)* *$M$ is $\sigma$-additive in the weak-$*$ topology, i.e. for countably many mutually disjoint sets $F_i$ one has*

$$\operatorname{tr}\left(\rho \sum_i M(F_i)\right) = \operatorname{tr}\left(\rho M\left(\bigcup_i F_i\right)\right) \quad \forall\ \rho \in \mathfrak{T}(\mathcal{H}),$$

*iii)* *$M$ is normalized, i.e. $M(\Omega) = \mathbb{1}_{\mathcal{H}}$.*

POVMs are often used in the theory of continuous measurements and were first described by Davies in 1976 (cf. [Dav76]). In the standard formulation of quantum mechanics this would simply correspond to observables being self adjoint operators, which implies that $\Omega \subset \mathbb{R}$ and $M$ being the spectral measure, s.t. any self adjoint operator $A$ can be written as

$$A = \int\limits_{\sigma(A)} x\, M(\mathrm{d}x).$$

In the case that we have a continuous spectrum, s.t. the integral doesn't reduce to a sum, we adapt the notation $M(\mathrm{d}x)$. The notation will be made clear when we define counting statistics.

*Remark:* The most common kind of POVM in this thesis will be given by a set of positive operators $M_i$ with $i \in \Omega$, summing to the identity. Proofs, however, will rely on the abstract definition. If all the operators involved are projections the POVM will be called a **projection valued measure (PVM)**.

## 2.8 Representation Theory of the CCR Algebra

As we will see, the Weyl CCR is going to play a major role in the upcoming analysis. Therefore we briefly collect some of the key aspects about its representation and look at a certain example of quantum channels related to this subject.

In the following discussion we need to distinguish between two similar notions, that of the CCR algebra and that of the CCR C*-algebra. We briefly discuss both approaches. Since both notions are related to the phase space we need some symplectic structure before.

**Definition 2.33.** *Let $(\Xi, \sigma)$ be a symplectic vector space like in definition 2.5. $(\Xi, \sigma)$ will be called the* ***phase space****, in analogy to the classical phase space $(\mathbb{R}^{2N}, \{\cdot,\cdot\})$ with the Poisson bracket.*

The Weyl CCR is now an irreducible representation of this symplectic space over some Hilbert space obeying certain properties.

**Definition 2.34.** *Let $W$ be an irreducible representation of the* ***Weyl commutation relations (Weyl CCR)*** *on $\Xi$ over a Hilbert space $\mathcal{H}$, i.e. a strongly continuous[5] map $W : (\Xi, \sigma) \longrightarrow \left(\mathcal{B}(\mathcal{H}), \|\cdot\|_{op}\right)$ into the unitary operators on $\mathcal{H}$, satisfying*

$$\left.\begin{aligned} W(\xi)W(\eta) &= e^{i\sigma(\xi,\eta)/2}\, W(\xi+\eta) \\ W(\xi)^\dagger &= W(-\xi) \end{aligned}\right\} \quad \forall\, \xi, \eta \in \Xi. \tag{2.26}$$

By the Stone-von Neumann theorem (cf. [Neu31], [Sto32]) we know that all irreducible representations of the Weyl CCR are unitary equivalent to the Schrödinger representation[6] which is given by the usual multiplication and differentiation operators $Q$ and $P$. The relations (2.26) then reduces to the well known equation $[Q, P] = i\,\mathbb{1}$.

A more abstract viewpoint is given by constructing a C*-algebra of elements obeying the CCR and then search for irreducible representations. The structure and notation will be based on [Pet90].

[5] Note that on the set of unitary operators the strong and weak operator topology coincide.

[6] After restricting the representation space, i.e. $L^2(\mathcal{H})$, to the dense subset of smooth functions with compact support.

**Definition 2.35.** *We denote the $C^*$-algebra of objects satisfying (2.26) over the symplectic space $(\Xi, \sigma)$ as*

$$CCR(\Xi, \sigma) = \{W(\xi) \,|\, \xi \in \Xi\}\,. \tag{2.27}$$

*We then look at (continuous) representations $\pi : CCR(\xi, \sigma) \longrightarrow \mathcal{B}(\mathcal{H})$ with some Hilbert space $\mathcal{H}$. Note that we are interested in strong or weak continuity, since norm continuity would be too restrictive as we will see now.*

**Proposition 2.36.** *Let $\eta \neq \xi \in \Xi$, then*

$$\|W(\eta) - W(\xi)\|_{op} \geq \sqrt{2}. \tag{2.28}$$

For the sake of presentation we omit the proof of this proposition and also state that the algebra $\mathrm{CCR}(\Xi, \sigma)$ is well defined and unique up to isomorphisms for every non-degenerate symplectic space.[7] Note that if $\Xi$ is a complex Hilbert space, then $\sigma(\xi, \eta) = \mathrm{Im}\,\langle \xi|\eta\rangle$ is a non-degenerate symplectic form on the $\mathbb{R}$-linear span of $\Xi$ as we have seen in definition 2.6.
As a conclusion of the proposition one sees that the one-parameter unitary group $t \longmapsto W(t\xi)$ is never norm continuous and that the $\mathrm{C}^*$-algebra $\mathrm{CCR}(\Xi, \sigma)$ is therefore never separable.

## States and Fields

We recapitulate Stone's theorem from [Sto32].

**Theorem 2.37** (Stone)**.** *Let $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous one-parameter unitary group. Then there exists a possibly unbounded self adjoint operator $A : \mathcal{D}_A \longrightarrow \mathcal{H}$, s.t.*

$$U_t = e^{itA} \qquad \forall\, t \in \mathbb{R}. \tag{2.29}$$

*Conversely, every (possibly unbounded) self adjoint operator induces a strongly continuous one-parameter unitary group by the same formula.*

[7] The interested reader may find the proof in [Pet90][Theorem 2.1, Proposition 2.2].

We can now look at a representation $\pi : \mathrm{CCR}(\Xi, \sigma) \longrightarrow \mathcal{B}(\mathcal{H})$ of the CCR algebra on a Hilbert space $(\mathcal{H}, \langle \cdot \,|\, \cdot \rangle)$. If the map

$$t \longmapsto \langle \pi(W(t\xi))x \,|\, y \rangle \qquad x, y \in \mathcal{H} \tag{2.30}$$

is continuous, then by Stone's theorem there exists a self-adjoint operator $B_\pi(\xi)$, s.t.

$$\pi(W(t\xi)) = \exp(itB_\pi(\xi)), \; t \in \mathbb{R}. \tag{2.31}$$

$B_\pi(\xi)$ is called **field operator** and since $W(t\xi)$ is not norm continous, $B_\pi(\xi)$ must be unbounded. The representation $\pi$ is called **regular**, if $B_\pi(\xi)$ exists in the test function space (with values in $\mathcal{H}$) for all $\xi$. If the representation $\pi$ is clear from context we omit the $\pi$ index in our notation and look at $W(\xi)$ as being already represented.

For now, let us fix a regular representation $\pi$ and compute the various commutation relations between Weyl operators and field operators. These can be obtained by formal differentiation w.r.t. some generic real parameter $t$.

**Theorem 2.38.** *For $\xi, \eta \in \Xi$ and $t \in \mathbb{R}$ the following relations hold on $\mathcal{D}_W$:*

*i)* $B_\pi(t\xi) = tB_\pi(\xi)$

*ii)* $B_\pi(\xi + \eta) = B_\pi(\xi) + B_\pi(\eta)$

*iii)* $[B_\pi(\xi), W(\eta)] = \sigma(\xi, \eta)W(\eta)$

*iv)* $[B_\pi(\xi), B_\pi(\eta)] = -i\sigma(\xi, \eta)\mathbb{1}$

*Here the forth equation is known as the usual CCR for field operators.*

*Proof.* Since the involved formulas are quite long, the proof is shifted to the appendix A.1. □

If we have a complex structure $J$ on $\Xi$ we can define a **creation operator** $B^+$ and an **annihiliation operator** $B^-$ via the field operator as

$$B_\pi^\pm(\xi) = \frac{1}{2}\left(B_\pi(J\xi) \mp iB_\pi(\xi)\right). \tag{2.32}$$

Is the complex structure given by the usual imaginary unit $i$, then equation (2.32) reduces to the familiar form of creation and annihilation operators commonly seen in advanced quantum theory books as

$$a(\xi) = \frac{1}{2}\left(iB_\pi(\xi) + B_\pi(i\xi)\right) \quad \text{and} \quad a^\dagger(\xi) = \frac{1}{2}\left(-iB_\pi(\xi) + B_\pi(i\xi)\right). \tag{2.33}$$

# Chapter 3

# Theory Of Continuous Measurement

We now have everything we need in order to define a limit process of discrete time evolutions. We split this analysis throughout this chapter into multiple smaller steps.

## 3.1 Concatenating Discrete Dilation Spaces

Let $\Theta \in \mathfrak{Z}([0,T])$ be an interval decomposition and $i \in I(\Theta)$ an arbitrary index. Furthermore let $\widehat{\mathbb{E}}_i \in \mathfrak{CP}(\mathcal{B}(\mathcal{H}))$ be a family of unital completely positive maps, so that $\widehat{\mathbb{E}}_i$ naturally describes a discrete time evolution. By Stinesprings factorization theorem 2.30 one can construct another family of maps for every time-step

$$\mathbb{E}_i \in \mathfrak{CP}(\mathcal{B}(\mathcal{K}_i \otimes \mathcal{H}), \mathcal{B}(\mathcal{H})),$$

such that $\mathbb{E}_i$ describes the evolution in the time-step $i$ and all possible measurements compatible with this evolution.[1] These maps are ultimately of the form

$$\mathbb{E}_i(X) = V_i^\dagger X V_i \quad \text{with} \quad V_i : \mathcal{H} \longrightarrow \mathcal{K}_i \otimes \mathcal{H}.$$

To extend this definition to the full interval $[0,T]$, one simply defines how to concatenate those maps:

[1]Note that we are going to assume $\mathcal{B}(\mathcal{K}_i \otimes \mathcal{H}) = \mathcal{B}(\mathcal{K}_i) \otimes \mathcal{B}(\mathcal{H})$.

**Definition 3.1.** *Let $\mathcal{H}, \mathcal{K}_1, \mathcal{K}_2$ be Hilbert spaces and $\mathbb{E}_i : \mathcal{B}(\mathcal{K}_i) \otimes \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$. We define the "concatenation" as*

$$\begin{aligned} \mathbb{E}_1 \circ \mathbb{E}_2 : \mathcal{B}(\mathcal{K}_1) \otimes \mathcal{B}(\mathcal{K}_2) \otimes \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}), \\ \mathbb{E}_1 \circ \mathbb{E}_2 := \mathbb{E}_1 \left( \mathbb{1}_{\mathcal{B}(\mathcal{K}_1)} \otimes \mathbb{E}_2 \right). \end{aligned} \tag{3.1}$$

*Likewise, one can define this at the level of dilation spaces with $V_i : \mathcal{H} \longrightarrow \mathcal{K}_i \otimes \mathcal{H}$ via*

$$\begin{aligned} V_2 \circ V_1 : \mathcal{H} \longrightarrow & \mathcal{K}_1 \otimes \mathcal{K}_2 \otimes \mathcal{H}, \\ V_2 \circ V_1 := & \left( \mathbb{1}_{\mathcal{K}_1} \otimes V_2 \right) V_1. \end{aligned} \tag{3.2}$$

*Iterating this arbitrary times directly leads to*

$$\prod_{i=1}^{n} \mathbb{E}_i : \mathcal{B}\left( \bigotimes_{i=1}^{n} \mathcal{K}_i \otimes \mathcal{H} \right) \longrightarrow \mathcal{B}(\mathcal{H}), \qquad \prod_{i=1}^{n} \mathbb{E}_i := \prod_{i=1}^{n-1} \mathbb{E}_i \circ \mathbb{E}_n \tag{3.3}$$

$$\prod_{i=1}^{n} V_i : \mathcal{H} \longrightarrow \bigotimes_{i=1}^{n} \mathcal{K}_i \otimes \mathcal{H}, \qquad \prod_{i=1}^{n} V_i := V_n \circ \prod_{i=1}^{n-1} V_i. \tag{3.4}$$

It is easy to see that since the iteration of maps is associative, it preserves the structure of Stinespring dilations in the following, natural way:

**Lemma 3.2.** *Given concatenated channels as in the definition before and for all $0 \leq i \leq n$ with $i \in I(\Theta)$ we have $\mathbb{E}_i(X) = V_i^\dagger X V_i$, then we have*

$$\prod_{i=1}^{n} \mathbb{E}_i(X) = \left( \prod_{i=1}^{n} V_i \right)^\dagger X \left( \prod_{i=1}^{n} V_i \right). \tag{3.5}$$

## 3.2 Limit Space Construction

The following construction relies on the work Bernhard Neukirchen did in his Ph.D. Thesis [Neu15]. His aim was to construct a Hilbert space for cMPS via a continuous version of dilation spaces. Starting with discrete time-steps, his work had three major assumptions:

i) A single time-step has either a single event or none.

ii) The quantum mechanical description of an event is independent of the length of the time-step.

iii) Obtaining multiple events just means to concatenate the single event construction.

Translating those assumptions in the mathematical background of quantum mechanics means, that i): the corresponding Hilbert space of a time-step consists of a "no-event" part, which is simply given by $\mathbb{C}$, and a non trivial part given by a separable Hilbert space $\mathcal{K}_i$. Part ii) means that the $\mathcal{K}_i$ do not depend upon the time-step $i$, so $\mathcal{K}_i = \mathcal{K}\ \forall i \in I(\Theta)$. Part iii) then simply tells us, that the Hilbert space of multiple events is just the tensor product of the single event Hilbert space.

The following figure intuitively shows the concatenation of discrete measurements w.r.t. an interval decomposition.

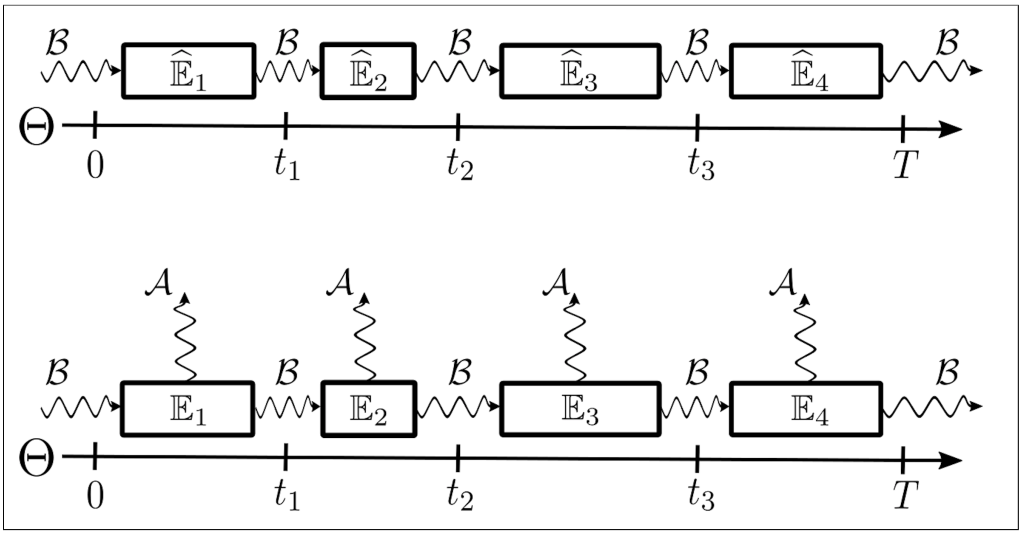



Figure 3.1: Composition of channels via an interval decomposition $\Theta \in \mathfrak{Z}([0,T])$.

As described above, the following choice of discrete Hilbert spaces seems natural to our assumptions: Let $(\mathfrak{Z}([0,T]), \subseteq)$ be the directed set of interval decompositions and let $\mathcal{K}$ be a separable Hilbert space. We define the Hilbert space corresponding to an interval decomposition $\Theta \in \mathfrak{Z}([0,T])$ as

$$\mathcal{K}_\Theta := \bigotimes_{i \in I(\Theta)} (\mathbb{C} \oplus \mathcal{K}). \tag{3.6}$$

Assume that $\Xi \in \mathfrak{Z}([0,T])$ is a finer interval decomposition than $\Theta$.
We denote by $\Xi|_i$ the interval decomposition of $[t_{i-1}, t_i]$ for $i \in I(\Theta)$ given by $\{t_j \in \Xi \mid t_{i-1} \leq t_j \leq t_i\}$. Then it is easy to see, that

$$\mathcal{K}_\Xi = \bigotimes_{i \in I(\Theta)} \mathcal{K}_{\Xi|_i}. \tag{3.7}$$

Let $|0\rangle$ be a distinguished "no-event" basis vector in $\mathbb{C}$ and $|\alpha\rangle \in \mathcal{K}$. In analogy to [Neu15] we use the following shorthand notation for a single event happening in one subinterval

$$|\alpha@k\rangle := \left( \bigotimes_{i<k \in I(\Theta)} \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \right) \otimes \begin{pmatrix} 0 \\ |\alpha\rangle \end{pmatrix} \otimes \left( \bigotimes_{i>k \in I(\Theta)} \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \right) \in \mathcal{K}_\Theta. \tag{3.8}$$

Using this notation we can define operators $J_i : \mathbb{C} \oplus \mathcal{K} \longrightarrow \mathcal{K}_{\Xi|_i}$ $\forall i \in I(\Theta)$ with $\tau_i = t_i - t_{i-1}$ via

$$J_i \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} := \bigotimes_{j \in I(\Xi|_i)} \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix}, \qquad J_i \begin{pmatrix} 0 \\ |\alpha\rangle \end{pmatrix} := \sum_{j \in I(\Xi|_i)} \sqrt{\frac{\tau_j}{\tau_i}} \, |\alpha@j\rangle \,, \tag{3.9}$$

s.t. one can define embeddings between interval decompositions

$$J_{\Xi\Theta} : \mathcal{K}_\Theta \longrightarrow \mathcal{K}_\Xi, \qquad J_{\Xi\Theta} := \bigotimes_{i \in I(\Theta)} J_i. \tag{3.10}$$

This way, one can compare different Hilbert spaces with each other.

**Lemma 3.3.** *$J_{\Xi\Theta}$ is an isometry between Hilbert spaces and $J_{\Lambda\Theta} = J_{\Lambda\Xi} J_{\Xi\Theta}$, i.e. the $J_{\Xi\Theta}$ form an inductive system of mappings.*

*Proof.* Since $J_{\Xi\Theta}$ is defined as the tensor product over the subintervals we restrict ourselves to one subinterval. The scalar product between two "no-event" embeddings is trivial, so we start with the action between "no-event" and "event" vectors:

$$\left\langle J_i \begin{pmatrix} 0 \\ |\alpha\rangle \end{pmatrix} \middle| J_i \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \right\rangle = \sum_{j\in I(\Xi|_i)} \sqrt{\frac{\tau_j}{\tau_i}} \left\langle \alpha@j \middle| \bigotimes_{k\in I(\Xi|_i)} \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \right\rangle$$
$$= \sum_{j\in I(\Xi|_i)} \sqrt{\frac{\tau_j}{\tau_i}} \underbrace{\left\langle \begin{pmatrix} \langle 0| \\ 0 \end{pmatrix} \middle| \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \right\rangle}_{=1} \cdots \underbrace{\left\langle \begin{pmatrix} 0 \\ \langle\alpha| \end{pmatrix} \middle| \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \right\rangle}_{=0 \text{ at j-th factor}} \cdots \underbrace{\left\langle \begin{pmatrix} \langle 0| \\ 0 \end{pmatrix} \middle| \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \right\rangle}_{=1}$$
$$= 0.$$

Now let $|\alpha\rangle, |\beta\rangle$ be arbitrary event vectors, then we have

$$\left\langle J_i \begin{pmatrix} 0 \\ |\alpha\rangle \end{pmatrix} \middle| J_i \begin{pmatrix} 0 \\ |\beta\rangle \end{pmatrix} \right\rangle = \sum_{j,k\in I(\Xi|_i)} \sqrt{\frac{\tau_j\tau_k}{\tau_i^2}} \langle \alpha@j|\beta@k\rangle = \sum_{j\in I(\Xi|_i)} \frac{\tau_j}{\tau_i} \langle \alpha@j|\beta@j\rangle$$
$$= \delta_{\alpha\beta} \frac{1}{\tau_i} \sum_{j\in I(\Xi|_i)} \tau_j = \delta_{\alpha\beta}.$$

Via extension by linearity the statement is proven. □

The obvious construction of a limit space is now a bit compelling, but the Hilbert space structure would get lost by simply taking the inductive limit with the aforementioned embeddings $J_{\Xi\Theta}$. The inductive limit space would only constitute a Pre-Hilbert space so one has to take its norm closure:

**Definition 3.4.** *Let $\Theta \subseteq \Xi \in \mathfrak{Z}([0,T])$. We define the* ***limit space*** *as the norm closure of the inductive limit of the system $(\mathcal{K}_\Theta, J_{\Xi\Theta})$, i.e.*

$$\mathcal{K}_{[0,T]} := \overline{J\text{-}\varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \mathcal{K}_\Theta}. \tag{3.11}$$

*$\mathcal{K}_{[0,T]}$ is a Hilbert space and we denote the canonical embedding into the limit space by $J_\Theta : \mathcal{K}_\Theta \longrightarrow \mathcal{K}_{[0,T]}$.*

## 3.3 Exponential Vectors and Fock Space Isomorphism

An interesting subset of this (quite abstract) limit space are the so-called exponential vectors, since they obey some interesting properties as we will see in this section. Because of their denseness in $\mathcal{K}_{[0,T]}$ and other reasons those vectors (defined in a similar fashion) are used quite often in quantum stochastic calculus (QSC).

However, before we can define the exponential vectors and check their convergence properties we need to get our head around a peculiar fact, i.e. the term $|\lambda(t_i)\rangle$ with $|\lambda\rangle \in L^2([0,T],\mathcal{K})$ is a priori not well defined because of the quotient space structure of $L^2$. We therefore restrict ourselves to the dense subspace of step functions first and then lift this notion to a definition on the full $L^2$ space.

Hence, let $|\lambda\rangle$ be a $\mathcal{K}$-valued step function w.r.t. to some interval decomposition $\Theta \in \mathfrak{Z}([0,T])$, i.e. $|\lambda\rangle \in \mathrm{Step}_{I(\Theta)}([0,T],\mathcal{K})$, which can be written as

$$|\lambda\rangle = \sum_{i\in I(\Theta)} \lambda_i \left|\chi_{[t_{i-1},t_i)}(t)\right\rangle \quad \text{with} \quad \lambda_i \in \mathcal{K}. \tag{3.12}$$

Those step functions are a dense subspace of $L^p([0,T],\mathcal{K})$ as seen in figure 2.2 for every $1 \le p < \infty$. Note that the step functions $\left\{f_i(t) = \frac{1}{\sqrt{\tau_i}}\chi_{[t_{i-1},t_i)} \mid i \in I\right\}$ are obviously orthonormal in $L^2([0,T],\mathbb{R})$ for some discrete index set $I$.

Using this orthonormal set we can define a discrete step function version of the exponential vectors acting on proper $L^2$-functions.

**Definition 3.5.** *Let $|\lambda\rangle \in L^2([0,T],\mathcal{K}) \cong L^2([0,T]) \otimes \mathcal{K}$ and $\Theta \in \mathfrak{Z}([0,T])$, we define the **discrete exponential vector** to be*

$$|e_\Theta(\lambda)\rangle := \bigotimes_{i\in I(\Theta)} \begin{pmatrix} 1 \\ (\langle f_i| \otimes \mathbb{1}_{\mathcal{K}})\,|\lambda\rangle \end{pmatrix} = \bigotimes_{i\in I(\Theta)} \begin{pmatrix} 1 \\ \sum_\alpha \langle f_i \otimes e_\alpha|\lambda\rangle\,|e_\alpha\rangle \end{pmatrix} \in \mathcal{K}_\Theta.$$

*In the latter equation we have written the scalar product in terms of some basis $\{|e_\alpha\rangle \in \mathcal{K} \mid \alpha = 1,\dots,\dim(\mathcal{K})\}$. Furthermore we denote the projection onto the step functions w.r.t. a given interval decomposition as*

$$P_\Theta : L^2([0,T],\mathcal{K}) \longrightarrow L^2([0,T],\mathcal{K})$$

$$P_\Theta(|\lambda\rangle) = \sum_{i\in I(\Theta)} \left(|f_i(t)\rangle\langle f_i(t)| \otimes \mathbb{1}_\mathcal{K}\right)|\lambda\rangle\,. \tag{3.13}$$

We therefore have $\mathrm{Step}_{I(\Theta)}([0,T],\mathcal{K}) \cap L^2([0,T],\mathcal{K}) = P_\Theta(L^2([0,T],\mathcal{K}))$. The Hilbert space embeddings $J_{\Xi\Theta}$, the outlined refinement limit is based upon, inherit a certain type of averaging operation between different time-steps, as we will see in the upcoming chapters. The following notational shortcuts will come out handy.

**Notation 3.6.** *From now on we will denote $\chi_{[t_{i-1},t_i)}$ simply as $\chi_i$. Furthermore we would like to use the tensor product structure of $L^p([0,T],\mathcal{K})$ and rewrite our functions $|\lambda\rangle \in L^2([0,T],\mathcal{K})$ in the following way:*

$$|\lambda\rangle = |\lambda_\mathbb{C}\rangle \otimes |\lambda_\mathcal{K}\rangle \quad \textit{with} \quad |\lambda_\mathbb{C}\rangle \in L^2([0,T],\mathbb{C}) \textit{ and } |\lambda_\mathcal{K}\rangle \in \mathcal{K}. \tag{3.14}$$

*Therefore, whenever we will encounter an $L^2$-function $|\lambda\rangle$ with values in some Hilbert space $\mathcal{K}$, we will write $\left|\overline{\lambda}_i\right\rangle \in \mathcal{K}$ defined as*

$$\begin{aligned}\left|\overline{\lambda}_i\right\rangle &= \frac{1}{\tau_i}\left(\langle\chi_i| \otimes \mathbb{1}_\mathcal{K}\right)|\lambda\rangle = \frac{1}{\tau_i}\int_0^T \lambda_\mathbb{C}(t)\chi_i\,\mathrm{d}t \otimes \mathbb{1}_\mathcal{K}\,|\lambda_\mathcal{K}\rangle \\ &= \frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} \lambda_\mathbb{C}(t)\mathrm{d}t\;|\lambda_\mathcal{K}\rangle\,, \end{aligned} \tag{3.15}$$

*i.e. we denote $\left|\overline{\lambda}_i\right\rangle \in \mathcal{K}$ as the average value of the $L^2([0,T])$ part of $|\lambda\rangle$ in the subinterval $[t_{i-1},t_i)$ multiplied with the $\mathcal{K}$-valued part.*

Using this notation we can rewrite the discrete exponential vectors in an easier fashion as

$$|e_\Theta(\lambda)\rangle := \bigotimes_{i\in I(\Theta)} \begin{pmatrix} 1 \\ \sqrt{\tau_i}\left|\overline{\lambda}_i\right\rangle \end{pmatrix}. \tag{3.16}$$

We now need to show well definedness of a limit object of those vectors.

**Theorem 3.7.** *The net* $\Theta \longmapsto J_\Theta e_\Theta(|\lambda\rangle) \in \mathcal{K}_{[0,T]}$ *converges strongly. We denote the inductive limit as*

$$|e(\lambda)\rangle = \varinjlim_{\Theta\in \mathfrak{Z}([0,T])} |e_\Theta(\lambda)\rangle .$$

*For all* $|\lambda\rangle, |\mu\rangle \in L^2([0,T],\mathcal{K})$ *those vectors obey the following* ***exponential property****:*

$$\langle e(\lambda)|e(\mu)\rangle := \varinjlim_{\Theta\in \mathfrak{Z}([0,T])} \langle e_\Theta(\lambda)|e_\Theta(\mu)\rangle = e^{\langle\lambda|\mu\rangle}. \tag{3.17}$$

Before we prove this theorem we need the following lemma.

**Lemma 3.8.** *Let* $\Theta \in \mathfrak{Z}([0,T])$ *be an arbitrary interval decomposition and* $\alpha \in P_\Theta\left(L^2([0,T],\mathbb{C})\right)$ *be a* $\mathbb{C}$*-valued step function. Then the following equation holds*

$$\varinjlim_{\Theta\in \mathfrak{Z}([0,T])} \prod_{i\in I(\Theta)} (1+\tau_i\alpha(t_i)) = \exp\left(\int_0^T \alpha(t)\mathrm{d}t\right). \tag{3.18}$$

*The integral on the right is meant to be the usual Riemann integral.*

*Proof.* First we need to rewrite the Riemann integral in its discrete form, i.e. we use the defining property of Riemann integrals for step functions

$$\varinjlim_{\Theta\in \mathfrak{Z}([0,T])} \sum_{i\in I(\Theta)} \tau_i\alpha(t_i) = \int_0^T \alpha(t)\mathrm{d}t,$$

as explained in equation (2.10).

Using this, taking the logarithm on both sides of equation (3.18) and analyzing this on some fixed $\Theta \in \mathfrak{Z}([0,T])$ we have

$$\log\left(\prod_{i\in I(\Theta)} (1+\tau_i\alpha(t_i))\right) = \sum_{i\in I(\Theta)} \log(1+\tau_i\alpha(t_i)) = \sum_{i\in I(\Theta)} \tau_i\alpha(t_i).$$

Since we consider a refinement limit we may assume $\tau_i \leq \varepsilon$ and therefore use

$$|\log(1+x) - x| = \left|\int_0^x \left(\frac{1}{1+y} - 1\right) \mathrm{d}y\right| = \left|\int_0^x \frac{-y}{1+y} \mathrm{d}y\right| \overset{y=tx}{=} \left|\int_0^1 \frac{tx^2}{1+tx} \mathrm{d}t\right|$$
$$= \left|x^2 \int_0^1 \frac{t}{1+tx} \mathrm{d}t\right| \leq |x^2| \int_0^1 \frac{t}{|1+tx|} \mathrm{d}t.$$

Substituting $x = \tau_i \alpha(t_i)$ we see that the denominator $|1 + t\,\tau_i \alpha(t_i)|$ could, for a specific choice of $\alpha(t_i)$ and $\tau_i$, equal zero which would make the integrand impossible to bound from above.

However, since we analyze this expression in the inductive limit where all $\tau_i \longrightarrow 0$, there is going to be an interval decomposition fine enough such that $|1 + t\,\tau_i \alpha(t_i)|$ can be bounded from below, i.e. never becomes zero in this limit, which is equivalent to $|\tau_i \alpha(t_i)| = \delta < 1$.

W.l.o.g. this interval decomposition will also be denoted by $\Theta \in \mathfrak{Z}([0,T])$. Furthermore let $C_\alpha = \sup\limits_{t\in[0,T]} |\alpha(t)|$ be an upper bound for $|\alpha|$, then we have

$$\Longrightarrow |\log(1 + \tau_i \alpha(t_i)) - \tau_i \alpha(t_i)| \leq \tau_i^2 |\alpha(t_i)^2| \int_0^1 \frac{t}{|1 + t\tau_i \alpha(t_i)|} \mathrm{d}t \leq C_\alpha^2 \frac{\tau_i\,\varepsilon}{1-\delta}.$$

Now, taking sums of those terms, and using the triangle inequality, we encounter an upper bound

$$\left|\log\left(\prod_{i\in I(\Theta)} (1 + \tau_i \alpha(t_i))\right) - \sum_{i\in I(\Theta)} \tau_i \alpha(t_i)\right| = \left|\sum_{i\in I(\Theta)} \log(1 + \tau_i \alpha(t_i)) - \tau_i \alpha(t_i)\right|$$
$$\leq \sum_{i\in I(\Theta)} |\log(1 + \tau_i \alpha(t_i)) - \tau_i \alpha(t_i)| \leq \frac{C_\alpha^2\,\varepsilon}{1-\delta} \sum_{i\in I(\Theta)} \tau_i = \frac{C_\alpha^2\,\varepsilon}{1-\delta} T$$

and hence, analyzing this in the inductive limit, we obtain

$$\varinjlim_{\Theta\in \mathfrak{Z}([0,T])} \left|\log\left(\prod_{i\in I(\Theta)} 1 + \tau_i \alpha\right) - \sum_{i\in I(\Theta)} \tau_i\,\alpha(t_i)\right| = 0.$$

We can therefore finally conclude

$$\lim_{\overrightarrow{\Theta \in \mathfrak{Z}([0,T])}} \prod_{i \in I(\Theta)} (1 + \tau_i \alpha) = \lim_{\overrightarrow{\Theta \in \mathfrak{Z}([0,T])}} \exp\left( \sum_{i \in I(\Theta)} \tau_i \, \alpha(t_i) \right) = \exp\left( \int_0^T \alpha(t) \mathrm{d}t \right),$$

which proves the statement. □

Using the denseness argumentation in figure 2.2, we can lift this lemma to piece wise continuous functions $\alpha : [0,T] \longrightarrow \mathbb{C}$, since those are the most general functions[2] for which the Riemann integral exists. The following corollary will be a direct consequence of this lemma.

We have now seen that products of the form $1 + \tau_i \alpha(t_i)$ converge to the exponential function in the inductive limit. This linear order of $\tau$ is a necessary condition for this limit to converge without being trivial. The following identity shows this mathematically

**Corollary 3.9.** *Let $\kappa > 0$, then*

$$\lim_{\overrightarrow{\Theta \in \mathfrak{Z}([0,T])}} \prod_{i \in I(\Theta)} (1 + \tau_i^{1+\kappa} \alpha(t_i)) = 1. \tag{3.19}$$

*Proof.* Completely analogous to the prior proof we would like to show that the following difference becomes small:

$$\begin{aligned} \left| \log\left( \prod_{i \in I(\Theta)} 1 + \tau_i^{1+\kappa} \alpha(t_i) \right) - \underbrace{\log(1)}_{=0} \right| &= \left| \sum_{i \in I(\Theta)} \log\big(1 + \tau_i^{1+\kappa} \alpha(t_i)\big) \right| \\ &\leq \sum_{i \in I(\Theta)} \left| \log\big(1 + \tau_i^{1+\kappa} \alpha(t_i)\big) \right|. \end{aligned}$$

We therefore concentrate on showing

$$\left| \log\big(1 + \tau_i^{1+\kappa} \alpha(t_i)\big) \right| \leq \varepsilon \longrightarrow 0.$$

[2]Taking denseness in $L^2$ into account.

To bound this logarithm we obtain for $|x| < 1$

$$|\log(1+x)| = \left| \int_0^x \frac{1}{1+y}\, \mathrm{d}y \right| = \left| x \int_0^1 \frac{1}{1+xt}\, \mathrm{d}t \right| \leq |x| \int_0^1 \frac{1}{|1+xt|}\, \mathrm{d}t \leq \frac{|x|}{1-|x|}.$$

The same argumentation as before (but now with $\delta = |x| = \left|\tau_i^{1+\kappa}\alpha(t_i)\right|$) guaranties that we can bound this integral by $(1-\delta)^{-1}$ for some interval decomposition fine enough. Using $\tau_i \leq \varepsilon$, $C_\alpha \geq |\alpha|$ and since taking the $(1+\kappa)$'s power is a strictly increasing function $\tau_i^{1+\kappa} = \tau_i\, \tau_i^\kappa$ holds, we conclude

$$\begin{aligned} \left| \sum_{i\in I(\Theta)} \log\big(1+\tau_i^{1+\kappa}\alpha(t_i)\big) \right| &\leq \sum_{i\in I(\Theta)} \tau_i^{1+\kappa} |\alpha(t_i)| \int_0^1 \frac{1}{\left|1+t\,\tau_i^{1+\kappa}\alpha(t_i)\right|}\, \mathrm{d}t \\ &\leq \sum_{i\in I(\Theta)} \tau_i\, \tau_i^\kappa\, C_\alpha \frac{1}{1-\delta} \leq \varepsilon^\kappa \frac{C_\alpha}{1-\delta} \sum_{i\in I(\Theta)} \tau_i = \varepsilon^\kappa \frac{T\, C_\alpha}{1-\delta}. \end{aligned}$$

Therefore the desired logarithm tends to zero which proves the statement after exponentiating both sides. □

It should be noted here, that if one would try to analyze the limit with $1-\kappa$ the resulting expression would not be well defined. Having this, we can now prove theorem 3.7.

*Proof of theorem 3.7.* We split the proof in two parts, we begin with the well-definedness.

**1. Convergence:** To check strong convergence we need to show that

$$\lim_{\Xi \gg \Theta} \left\| |e_\Xi(\lambda)\rangle - J_{\Xi\Theta}\, |e_\Theta(\lambda)\rangle \right\| = 0 \quad \text{for all } |\lambda\rangle \in L^2([0,T],\mathcal{K}).$$

Since we are calculating the norm on Hilbert spaces we can look at the norm square and use $\|a-b\|^2 = \|a\|^2 + \|b\|^2 - 2\,\mathrm{Re}\,\langle a|b\rangle$. Since $J_{\Xi\Theta}$ are isometries, the first two resulting terms are the same, i.e. $\left\||e_\Theta(\lambda)\rangle\right\|^2 = \left\|J_{\Xi\Theta}\,|e_\Theta(\lambda)\rangle\right\|^2$. We begin with those terms

$$\left\| |e_\Xi(\lambda)\rangle \right\|^2 = \langle e_\Xi(\lambda)|e_\Xi(\lambda)\rangle = \left\langle \bigotimes_{i\in I(\Xi)} \begin{pmatrix} 1 \\ \sqrt{\tau_i}\,|\overline{\lambda}_i\rangle \end{pmatrix} \middle| \bigotimes_{i\in I(\Xi)} \begin{pmatrix} 1 \\ \sqrt{\tau_i}\,|\overline{\lambda}_i\rangle \end{pmatrix} \right\rangle$$

$$= \prod_{i\in I(\Xi)} \left\langle \begin{pmatrix} 1 \\ \sqrt{\tau_i}\,|\overline{\lambda}_i\rangle \end{pmatrix} \middle| \begin{pmatrix} 1 \\ \sqrt{\tau_i}\,|\overline{\lambda}_i\rangle \end{pmatrix} \right\rangle = \prod_{i\in I(\Xi)} \left(1 + \tau_i \left\langle \overline{\lambda}_i | \overline{\lambda}_i \right\rangle\right).$$

The scalar product $2\,\mathrm{Re}(\langle e_\Xi(\lambda)|J_{\Xi\Theta}e_\Theta(\lambda)\rangle)$ is a bit harder. It will be helpful to note that tensor product terms of the following form will cancel, since the j-th site will always contribute a zero when taking the scalar product of both terms, i.e.

$$\left\langle \bigotimes_{i\in I(\Xi)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| \sum_{j\in I(\Xi)} |\alpha@j\rangle \right\rangle = 0 \quad \forall\, |\alpha\rangle \in \mathcal{K}.$$

Using this we have

$$\langle e_\Xi(\lambda)|J_{\Xi\Theta}e_\Theta(\lambda)\rangle = \left\langle \bigotimes_{k\in I(\Xi)} \begin{pmatrix} 1 \\ \sqrt{\tau_k}\,|\overline{\lambda}_k\rangle \end{pmatrix} \middle| \left(\bigotimes_{i\in I(\Theta)} J_i\right) \left(\bigotimes_{i\in I(\Theta)} \begin{pmatrix} 1 \\ \sqrt{\tau_i}\,|\overline{\lambda}_i\rangle \end{pmatrix}\right) \right\rangle$$

$$= \left\langle \bigotimes_{i\in I(\Theta)} \bigotimes_{k\in I(\Xi|_i)} \begin{pmatrix} 1 \\ \sqrt{\tau_k}\,|\overline{\lambda}_k\rangle \end{pmatrix} \middle| \bigotimes_{i\in I(\Theta)} \left( \bigotimes_{k\in I(\Xi|_i)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{k\in I(\Xi|_i)} \sqrt{\tau_k}\,|\overline{\lambda}_i@k\rangle \right) \right\rangle$$

$$= \prod_{i\in I(\Theta)} \left\langle \bigotimes_{k\in I(\Xi|_i)} \begin{pmatrix} 1 \\ \sqrt{\tau_k}\,|\overline{\lambda}_k\rangle \end{pmatrix} \middle| \bigotimes_{k\in I(\Xi|_i)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{k\in I(\Xi|_i)} \sqrt{\tau_k}\,|\overline{\lambda}_i@k\rangle \right\rangle$$

$$= \prod_{i\in I(\Theta)} \prod_{k\in I(\Xi|_i)} \left[ \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} 0 \\ \sqrt{\tau_k}\,|\overline{\lambda}_k\rangle \end{pmatrix} \middle| \begin{pmatrix} 0 \\ \sqrt{\tau_k}\,|\overline{\lambda}_k\rangle \end{pmatrix} \right\rangle \right]$$

$$= \prod_{i\in I(\Xi)} \left(1 + \tau_i \left\langle \overline{\lambda}_i | \overline{\lambda}_i \right\rangle\right).$$

Therefore the desired norm difference becomes

$$\lim_{\Xi \gg \Theta} \left\| |e_\Xi(\lambda)\rangle - J_{\Xi\Theta}\,|e_\Theta(\lambda)\rangle \right\| = 0$$

for all $|\lambda\rangle \in L^2([0,T],\mathcal{K})$, which proves the strong convergence. Since those vectors are well defined in the inductive limit, we can now prove the second part of the lemma.

**2. Exponential property:** We restrict ourselves again to step functions $|\lambda\rangle$ and $|\mu\rangle$ which canonically induce interval decompositions $\Theta_\lambda$ and $\Theta_\mu$, i.e. $|\lambda\rangle \in P_{\Theta_\lambda}\left(L^2([0,T],\mathcal{K})\right)$ and for $|\mu\rangle$ respectively.

Since we are dealing with an inductive limit we can assume that all interval decompositions $\Theta$ we construct our limit upon are finer, i.e. $\Theta \subseteq \Theta_\lambda \cup \Theta_\mu$. Note that on those interval decompositions the $L^2([0,T])$ tensor factors are locally constant:

$$|\lambda\rangle = |\lambda_\mathbb{C}\rangle \otimes |\lambda_\mathcal{K}\rangle = \sum_{j\in I(\Theta)} \lambda_j \left|\chi_j\right\rangle \otimes |\lambda_\mathcal{K}\rangle\,; \quad \lambda_j \in \mathbb{C}$$

$$\Longrightarrow \left|\overline{\lambda}_i\right\rangle = \frac{1}{\tau_i}\left(\langle\chi_i| \otimes \mathbb{1}_\mathcal{K}\right)|\lambda\rangle = \frac{1}{\tau_i}\sum_{j\in I(\Theta)} \lambda_j \underbrace{\left\langle\chi_i|\chi_j\right\rangle}_{=\tau_i\delta_{ij}} \otimes |\lambda_\mathcal{K}\rangle = \lambda_i\,|\lambda_\mathcal{K}\rangle\,.$$

Therefore we have that

$$\begin{aligned}\langle\lambda|\mu\rangle &= \langle\lambda_\mathbb{C}|\mu_\mathbb{C}\rangle\,\langle\lambda_\mathcal{K}|\mu_\mathcal{K}\rangle = \langle\lambda_\mathcal{K}|\mu_\mathcal{K}\rangle \sum_{i\in I(\Theta)} \int_{t_{i-1}}^{t_i} \lambda_\mathbb{C}(t)^*\mu_\mathbb{C}(t)\,\mathrm{d}t \\ &= \langle\lambda_\mathcal{K}|\mu_\mathcal{K}\rangle \sum_{i\in I(\Theta)} \tau_i\lambda_i^*\mu_i. \end{aligned} \tag{3.20}$$

Using lemma 3.8 we see

$$\begin{aligned}\langle e(\lambda)|e(\mu)\rangle &\overset{\text{Def.}}{=} \varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \langle e_\Theta(\lambda)|e_\Theta(\mu)\rangle = \varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \prod_{i\in I(\Theta)} \left(1+\tau_i\left\langle\overline{\lambda}_i|\overline{\mu}_i\right\rangle\right) \\ &= \varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \prod_{i\in I(\Theta)} \left(1+\tau_i\lambda_i^*\mu_i\,\langle\lambda_\mathcal{K}|\mu_\mathcal{K}\rangle\right) \\ &\overset{(3.18)}{=} \varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \exp\left(\sum_{i\in I(\Theta)} \tau_i\lambda_i^*\mu_i\,\langle\lambda_\mathcal{K}|\mu_\mathcal{K}\rangle\right) \overset{(3.20)}{=} e^{\langle\lambda|\mu\rangle}.\end{aligned}$$

Using the denseness of step functions in $L^2([0,T],\mathcal{K})$ completes the proof. □

To see that the exponential vectors do not constitute a small subset of $\mathcal{K}_{[0,T]}$ is of key importance. In fact their $\mathbb{C}$ span is all of $\mathcal{K}_{[0,T]}$. More formally:

**Theorem 3.10.** *The $\mathbb{Q}$ linear span of exponential vectors $|e_\Theta(\lambda)\rangle$ is dense in $\mathcal{K}_\Theta$. Furthermore we have that the $\mathbb{C}$ linear span of $|e_\Theta(\lambda)\rangle$ equals $\mathcal{K}_\Theta$.*

*Proof.* Since $\mathcal{K}_\Theta$ and $|e_\Theta(\lambda)\rangle$ have the same tensor product structure it is sufficient to show denseness on one factor. That means we need to show that

$$\mathrm{span}_{\mathbb{Q}} \left\{ 1 \oplus \sqrt{\tau_i} \left| \overline{\lambda}_i \right\rangle \right\} = \mathbb{C} \oplus \mathcal{K},$$

which would certainly not be true if the image of $|\lambda\rangle$ at the i-th subinterval would not span all of $\mathcal{K}$, but since $\mathcal{K}$ is constructed as the minimal dilation space, coming from Stinesprings factorization theorem, this can not be the case. Therefore the $\mathbb{Q}$ linear span of $1 \oplus \sqrt{\tau_i} \left| \overline{\lambda}_i \right\rangle$ is dense in $\mathbb{C} \oplus \mathcal{K}$, since $\mathbb{Q}$ is dense in $\mathbb{C}$ and the theorem is proven. □

The existence of exponential vectors is a strong hint for our limit space to have some structural similarities to Fock space. Those kind of isomorphisms are heavily used in quantum stochastic calculus. We shall briefly explain the basic idea:

One can show, as seen in [Neu15][Theorem 6.22, Theorem 6.23], that the limit space $\mathcal{K}_{[0,T]}$ is isomorphic to the bosonic/fermionic Fock spaces, i.e.

$$\mathcal{K}_{[0,T]} \cong \overline{\bigoplus_{n=0}^{\infty} L^2(\Delta_n, \mathcal{K}) \otimes \mathcal{K}^{\otimes n}} = \Gamma_\pm(L^2([0,T], \mathcal{K})), \tag{3.21}$$

with the Fock space functors $\Gamma_\pm$ and the set of ordered n-tuples $\Delta_n$. The isomorphism is meant to be in the sense, that there exists a canonical unitary equivalence on the n-particle level. A more detailed analysis of this isomorphism can be found in [Sie]. We will work with the Bose Fock space, rather then with the Fermi Fock space, since we already analyzed the CCR and not the CAR. A fermionic analysis is, nevertheless, quite an interesting field of research.

## 3.4 Operators on $\mathcal{K}_\Theta$

Since $\mathcal{K}_\Theta$ is just a finite tensor product of Hilbert spaces we can define operators on it. We will usually denote operators in a special form, using a crucial but necessary approximation.

**Definition 3.11.** *Let $c \in \mathbb{C}, |\lambda\rangle, |\mu\rangle \in \mathcal{K}$ and $U \in \mathcal{B}(\mathcal{K})$, then we denote*

$$\begin{pmatrix} c & \langle\lambda| \\ |\mu\rangle & U \end{pmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{K}) \quad \text{with} \quad (a, |\nu\rangle) \longmapsto (ac + \langle\lambda|\nu\rangle, a|\mu\rangle + U|\nu\rangle). \tag{3.22}$$

*We now assume that* $\mathcal{B}(\mathcal{K}_\Theta) = \mathcal{B}\left(\bigotimes_{i \in I(\Theta)} \mathbb{C} \oplus \mathcal{K}\right) \approx \bigotimes_{i \in I(\Theta)} \mathcal{B}(\mathbb{C} \oplus \mathcal{K})$. *One can therefore approximate every operator $X$ on $\mathcal{K}_\Theta$ via*

$$X \approx \sum_{n=1}^{N} \bigotimes_{i \in I(\Theta)} U_{i,n} \quad \text{with} \quad U_{i,n} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{K}) \text{ and } N \in \mathbb{N}. \tag{3.23}$$

We will always assume that every operator of physical interest will have such a tensor product form, since anything else but this would seem pathological. A class of operators with special importance are those with an operator $A \in \mathcal{B}(\mathbb{C} \oplus \mathcal{K})$ in a single time-step.

**Definition 3.12.** *We extend our "@i" notation in the following way to operators acting on bosons*

$$(A@i) = \left(\bigotimes_{k<i \in I(\Theta)} \mathbb{1}_{\mathbb{C}\oplus\mathcal{K}}\right) \otimes A \otimes \left(\bigotimes_{k>i \in I(\Theta)} \mathbb{1}_{\mathbb{C}\oplus\mathcal{K}}\right) \in \mathcal{B}(\mathcal{K}_\Theta) \tag{3.24}$$

*and on fermions respectively*

$$(A@i)_- = \left(\bigotimes_{k<i \in I(\Theta)} \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1}_{\mathcal{K}} \end{pmatrix}\right) \otimes A \otimes \left(\bigotimes_{k>i \in I(\Theta)} \mathbb{1}_{\mathbb{C}\oplus\mathcal{K}}\right) \in \mathcal{B}(\mathcal{K}_\Theta). \tag{3.25}$$

One is now able to define nets $\Theta \longmapsto X_\Theta \in \mathcal{B}(\mathcal{K}_\Theta)$ and analyze their convergence properties. Later on we will see necessary conditions for the data $(c, |\lambda\rangle, |\mu\rangle, U)$ of those operators to converge w.r.t. certain topologies.

# Chapter 4

# First Definitions of FCS and cMPS

In this chapter we will shortly collect the definitions of "Finitely Correlated States" (FCS) and "continuous Matrix Product States"(cMPS) in their original form. We will rewrite the FCS in the continuous measurement formalism which will motivate a strategy for redefining cMPS, using the inductive limits already discussed.

Note that the term "Matrix Product State" (MPS) is taken synonymously for FCS, since MPS are just FCS written w.r.t. specific choice of basis.

## 4.1 Finitely Correlated States (FCS)

FCS were first defined in [FNW92] as a model for interacting quantum spin chains. We start by recapitulating this first definition.

**Definition 4.1.** *Let $\mathcal{A}$ be the observable algebra for a fixed quantum system. $\mathcal{A}$ is a $C^*$-algebra with identity $\mathbb{1}_{\mathcal{A}}$. If this algebra is finite-dimensional then it is going to be the algebra of complex $d \times d$ matrices $\mathcal{M}_d$.*

For each $n \in \mathbb{Z}$ consider an isomorphic copy $\mathcal{A}_{\{n\}}$ of $\mathcal{A}$ and define $A_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{A}_{\{x\}}$ for every finite subset $\Lambda \in \mathbb{Z}$ where the tensor product refers to the minimal $C^*$-tensor product (cf. [Tak02]).

Now one can obtain the **chain algebra** $\mathcal{A}_\mathbb{Z}$ as a $C^*$-inductive limit of the algebras $\mathcal{A}_{\Lambda'}$ with $\Lambda' \subset \Lambda$ finite and the homomorphisms $\mathcal{A}_{\Lambda'} \hookrightarrow \mathcal{A}_\Lambda$ are given by tensoring $A \in \mathcal{A}_{\Lambda'}$ with $\bigotimes_{x \in \Lambda \setminus \Lambda'} \mathbb{1}_{\mathcal{A}_{\{x\}}}$. $\mathbb{Z}$ has a trivial action on $\mathcal{A}_\mathbb{Z}$ given by translation. The translation invariant states on the chain algebra are denoted by $\mathcal{T}(\mathcal{A})$.

The following construction is the original definition of FCS in [FNW92].

**Proposition 4.2.** *Let $\mathcal{A}$ be a $C^*$-algebra with unit, and let $\omega \in \mathcal{T}(\mathcal{A})$ be a translation invariant state on the chain algebra $\mathcal{A}_{\mathbb{Z}}$. Then the following are equivalent:*

*i) The set of functionals $\Phi : \mathcal{A}_{\mathbb{N}} \longrightarrow \mathbb{C}$ of the form*

$$\Phi(A_1 \otimes \cdots \otimes a_n) = \omega(X \otimes A_1 \otimes \cdots \otimes A_n), \tag{4.1}$$

*with $X \in \mathcal{A}_{\mathbb{Z}\setminus\mathbb{N}}$ generates a finite-dimensional linear subspace in the dual of $\mathcal{A}_{\mathbb{N}}$.*

*ii) There exists a finite-dimensional vector space $\mathcal{B}$, a linear map $\mathbb{E} : A \in \mathcal{A} \longmapsto \mathbb{E}_A \in L(\mathcal{B}, \mathcal{B})$, an element $e \in \mathcal{B}$, and a linear functional $\rho \in \mathcal{B}'$, such that $\rho \circ \mathbb{E}_1 = \rho$, $\mathbb{E}_1(e) = e$, and for $n \in \mathbb{Z}, m \in \mathbb{N}$ and $A_i \in \mathcal{A}_{\{i\}} \cong \mathcal{A}$:*

$$\omega(A_n \otimes \cdots \otimes A_{n+m}) = \rho(e)^{-1} \rho \circ \mathbb{E}_{A_n} \circ \cdots \circ \mathbb{E}_{A_{n+m}}(e). \tag{4.2}$$

*If in ii) $\mathcal{B}$ is chosen as minimal in the sense that*

$$span \left\{ \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n}(e) \;\middle|\; n \in \mathbb{N}, A_1, \ldots, A_n \in \mathcal{A} \right\} = \mathcal{B},$$

*and*

$$span \left\{ \rho \circ \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n}(e) \;\middle|\; n \in \mathbb{N}, A_1, \ldots, A_n \in \mathcal{A} \right\} = \mathcal{B}',$$

*then $\mathcal{B}, \mathbb{E}, \rho$ and $e$ are determined by $\omega$ up to a linear isomorphism.*

**Definition 4.3.** *If the equivalent conditions of proposition 4.2 are satisfied, $\omega$ will be called the* ***finitely correlated state generated*** *by $(\mathbb{E}, \rho, e)$.*

This original definition of FCS can also be put into the formalism we have encountered in the construction of continuous dilation spaces. Defining $\mathcal{A}_i = \mathcal{B}(\mathcal{K}_i)$ as the "chain" of environments, one can easily see the connection to the definition above. Note the finite dimension of $\mathcal{B}$.

**Definition 4.4.** *Let $\mathcal{H}$ be a Hilbert space and $\rho \in \mathfrak{T}(\mathcal{H})$ be a state on $\mathcal{B}(\mathcal{H})$, i.e. positive and of trace one. Also let $\mathcal{K}_i$ be a separable Hilbert space for all $1 \leq i \leq n$ and let $\mathbb{E}_i : \mathcal{B}(\mathcal{K}_i \otimes \mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a channel, i.e. completely positive, unital and normal.*

*A* ***finitely correlated state*** *$\omega$ belonging to the tuple $(\mathbb{E}_{i=1}^n, \rho)$ is given by:*

$$\omega : \bigotimes_{i=1}^{n} \mathcal{B}(\mathcal{K}_i) \longrightarrow \mathbb{C}, \qquad X \longmapsto \operatorname{tr}\left(\rho \left(\prod_{i=1}^{n} \mathbb{E}_i\right)(X \otimes \mathbb{1}_{\mathcal{H}})\right). \tag{4.3}$$

*That this is an equivalent definition follows from simple tensor product identities and from the fact that $\mathcal{B}(\mathcal{K})$ constitutes a $C^*$-algebra. Furthermore, if there exists a family of isometries $V_i : \mathcal{H} \longrightarrow \mathcal{K}_i \otimes \mathcal{H}$ for $1 \leq i \leq n$ s.t.*

$$\mathbb{E}_i : \mathcal{B}(\mathcal{K}_i \otimes \mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}), \qquad X \longmapsto V_i^* X V_i, \tag{4.4}$$

*the FCS is called* ***purely generated*** *and if the $\mathcal{K}_i = \mathcal{K}\ \forall\ i$, s.t. $\mathbb{E}_i = \mathbb{E}$ then it is called* ***translation invariant****.*

The C$^*$-algebra $\mathcal{B}$ was assumed to be finite-dimensional which makes (in our formalism) $\mathcal{K}_i$ finite-dimensional, too.

Since we assumed that $\mathcal{K} = \mathcal{K}_i$ right from the start by assumption number two and our analysis relies on the Stinespring theorem, we will see that the FCS (and later cMPS) arising from the continuous measurement setting are automatically purely generated and translation invariant.

## 4.2 Continuous Matrix Product States (cMPS)

cMPS were first defined in 2010 (cf. [VC10]) as one-dimensional quantum fields and a "continuous analogue" of MPS/FCS. This analogy was, however, more heuristically, then rigorous. The cMPS quantum field is defined as

$$|\chi\rangle = \operatorname{tr}_{\text{aux}}\left[\mathcal{P}\ \exp\left(\int_0^L \mathrm{d}x \left[Q(x) \otimes \mathbb{1} + R(x) \otimes \hat{\psi}^\dagger(x)\right]\right)\right]|\Omega\rangle \tag{4.5}$$

$$= \sum_{n=0}^{\infty} \int\limits_{0<x_1<\cdots<x_n<L} \mathrm{d}x_1 \ldots \mathrm{d}x_n \phi_n \hat{\psi}^\dagger(x_1) \ldots \hat{\psi}^\dagger(x_n) \left|\Omega\right\rangle \tag{4.6}$$

where

$$\phi_n = \mathrm{tr}_{\mathrm{aux}} \left[ u_Q(x_1, 0)\, R\, u_Q(x_2, x_1)\, R \cdots R\, u_Q(L, x_n) \right],$$
$$u_Q(y, x) = \mathcal{P} \exp \left( \int\limits_x^y Q(x) \mathrm{d}x \right)$$

and the traces are taken w.r.t. a finite-dimensional auxiliary Hilbert space. The matrix valued functions $Q$ and $R$ act on this auxiliary space, $\mathcal{P}\exp$ denotes the path ordered exponential and $\psi$ are field operators obeying the canonical commutation relations with the unique vacuum state $\left|\Omega\right\rangle$ defined via $\hat{\psi}\left|\Omega\right\rangle = 0$.

To see how this definition is comparable to the cMPS defined at the end of this thesis is going to be an interesting outlook, especially if one analyzes Fermions and has to impose Pauli's exclusion principle somewhere.

## Correspondence to cMPS using Continuous Measurements

Searching for a rigorous limit of MPS/FCS seems easier, when they are defined in the form

$$X \longmapsto \mathrm{tr} \left( \rho \left( \prod_{i=1}^{n} \mathbb{E}_i \right) (X \otimes \mathbb{1}_{\mathcal{H}}) \right),$$

since one could take an interval decomposition $\Theta \in \mathfrak{Z}([0, T])$ and analyze the convergence properties of the net $\Theta \longmapsto \prod\limits_{i \in I(\Theta)} \mathbb{E}_i$.

Defining a cMPS to be the refinement limit w.r.t. those interval decompositions and those "infinitely fine" concatenations seems to be a very natural idea and ultimately easier then coping with path ordered quantum fields. Construction of a "continuous measurement" field, arising from a "limit channel" and showing its well definedness is going to be subject of the following chapters.

# Chapter 5

# Continuous Weyl and Field Operators

We have seen how we approximate operators on each of the discrete spaces $\mathcal{K}_\Theta$ in chapter 3. We have also seen how to break down the convergence conditions for different topologies involving the limit space into terms only analyzing discrete object with each other in chapter 2. We will combine both notions to obtain necessary criteria for operators on $\mathcal{K}_{[0,T]}$ to be well defined.

## 5.1 Limit Space Operators

Since we perform a refinement limit of our Hilbert spaces $\mathcal{K}_\Theta \xrightarrow{J_{\Xi\Theta}} \mathcal{K}_\Xi$ we need to analyze how we come from $\mathcal{B}(\mathcal{K}_\Theta)$ to $\mathcal{B}(\mathcal{K}_\Xi)$. In chapter 2 we say that our limit constructions is mostly based on norm differences of the following form

$$\lim_{\Xi \gg \Theta} \left\| W_\Theta - J^\dagger_{\Xi\Theta} W_\Xi J_{\Xi\Theta} \right\|_{\mathrm{op}} = 0,$$

like here for norm convergence of operator nets. The embeddings $J_{\Xi\Theta}$ are constructed as a special kind of "averaging operation" on the operator level. To see this we will proof the following theorem.

**Theorem 5.1.** *Let $\Theta \subseteq \Xi \in \mathfrak{Z}([0,T])$. We set $|\lambda_j\rangle, |\nu_j\rangle \in \mathcal{K}, 0 \neq c_j \in \mathbb{C}$ and $O_j \in \mathcal{B}(\mathcal{K})$ for every $j \in I(\Xi)$. Then the operators*

$$W_j = \begin{pmatrix} c_j & \sqrt{\tau_j}\,\langle\nu_j| \\ \sqrt{\tau_j}\,|\lambda_j\rangle & O_j + \tau_j \frac{|\lambda_j\rangle\langle\nu_j|}{c_j} \end{pmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{K}) \tag{5.1}$$

*are of the following form after embedding*

$$\bigotimes_{i \in I(\Theta)} \widetilde{W}_i = J^\dagger_{\Xi\Theta} \left( \bigotimes_{j \in I(\Xi)} W_j \right) J_{\Xi\Theta}, \tag{5.2}$$

*where*

$$\widetilde{W}_i = \left( \prod_{j \in I(\Xi|_i)} c_j \right) \begin{pmatrix} 1 & \sqrt{\tau_i}\,\mathfrak{M}_i^\Xi\left(\frac{\langle\nu|}{c}\right) \\ \sqrt{\tau_i}\,\mathfrak{M}_i^\Xi\left(\frac{|\lambda\rangle}{c}\right) & \mathfrak{M}_i^\Xi\left(\frac{O}{c}\right) + \tau_i\,\mathfrak{M}_i^\Xi\left(\frac{|\lambda\rangle}{c}\right)\mathfrak{M}_i^\Xi\left(\frac{\langle\nu|}{c}\right) \end{pmatrix}. \tag{5.3}$$

*Here we have used the short notation*

$$\mathfrak{M}_i^\Xi(X) := \sum_{j \in I(\Xi|_i)} \frac{\tau_j}{\tau_i} X_j \tag{5.4}$$

*for averaging over the subinterval* $\Xi|_i$.

*Proof.* For the sake of presentation, the proof is shifted to Appendix A.2. □

The lower right matrix element of $W_j$ had this special form such that the embedded operator can be written completely in terms of averaging operations. The special $\tau$ dependence in this theorem will become clear in the following discussion.

Whenever an operator is going to be embedded, we will refer to this theorem, even though the starting operator is not exactly of the form in equation (5.1). The proof works analogously for any other form of $W_j$.

We now want to analyze which orders of $\tau$ are necessary to get a well defined operator in the inductive limit and which does not become trivial. At first we check which orders of $\tau$ in each matrix element would lead the operator to converge to the identity and hence can be ignored.

**Theorem 5.2.** *Let $\kappa > 0$ a positive real number, $|\lambda_1\rangle, |\lambda_2\rangle : [0,T] \longrightarrow \mathcal{K}, c : [0,T] \longrightarrow \mathbb{C}$ and $U : [0,T] \longrightarrow \mathcal{B}(\mathcal{K})$ bounded functions, i.e. there exists a $C_{\lambda_i} \in \mathbb{R}$ s.t. $\sup_{t\in[0,T]} \|\lambda_i(t)\| \leq C_{\lambda_i}$ and analogously for $c(t)$ and $U(t)$.*

*Then the following net of operators converges in weak-* topology to the identity*

$$W_\Theta^{id} = \bigotimes_{i\in I(\Theta)} \begin{pmatrix} 1+\tau_i^{1+\kappa}c(t_i) & \tau_i^{\frac{1}{2}+\kappa}\langle\lambda_1(t_i)| \\ \tau_i^{\frac{1}{2}+\kappa}|\lambda_2(t_i)\rangle & \mathbb{1}+\tau_i^{\kappa}U(t_i) \end{pmatrix} \in \mathcal{B}(\mathcal{K}_\Theta). \tag{5.5}$$

*Proof.* Let us recall that since our net maps into a reflexive Banach space, the weak and weak-* topologies coincide.

Now $W_\Theta \in \mathcal{B}(\mathcal{K}_\Theta)$ is $\widetilde{\text{weak}}$ convergent, and therefore by theorem 2.20 weak convergent, iff $|\varphi_\Theta\rangle, |\psi_\Theta\rangle$ being Cauchy implies that $\langle\varphi_\Theta|W_\Theta\psi_\Theta\rangle$ is Cauchy (and therefore convergent). Now since the linear span of exponential vectors is dense in $\mathcal{K}_\Theta$ (see theorem 3.10) it is sufficient to check $\widetilde{\text{weak}}$ convergence on this subset. We also restrict the domain of the exponential vectors to step functions.

So weak convergence of $W_\Theta$ to the identity is equivalent to showing

$$\lim_{\overrightarrow{\Theta\in\mathfrak{Z}([0,T])}} \left\langle e_\Theta(\nu)\middle|\left(W_\Theta^{\text{id}}-\mathbb{1}\right)e_\Theta(\mu)\right\rangle = 0 \quad \forall\,|\nu\rangle, |\mu\rangle \in P_\Theta\left(L^2([0,T],\mathcal{K})\right).$$

We calculate

$$\begin{aligned}
&\left\langle e_\Theta(\nu)\middle|W_\Theta^{\text{id}}e_\Theta(\mu)\right\rangle \\
&= \prod_{i\in I(\Theta)} \left\langle \begin{pmatrix} 1 \\ \sqrt{\tau_i}\langle\overline{\nu}_i| \end{pmatrix} \middle| \begin{pmatrix} 1+\tau_i^{1+\kappa}c(t_i) & \tau_i^{\frac{1}{2}+\kappa}\langle\lambda_1(t_i)| \\ \tau_i^{\frac{1}{2}+\kappa}|\lambda_2(t_i)\rangle & \mathbb{1}+\tau_i^{\kappa}U(t_i) \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{\tau_i}|\overline{\mu}_i\rangle \end{pmatrix} \right\rangle \\
&= \prod_{i\in I(\Theta)} \left\langle \begin{pmatrix} 1 \\ \sqrt{\tau_i}\langle\overline{\nu}_i| \end{pmatrix} \middle| \begin{pmatrix} 1+\tau_i^{1+\kappa}c(t_i)+\tau_i^{1+\kappa}\langle\lambda_1(t_i)|\overline{\mu}_i\rangle \\ \tau_i^{\frac{1}{2}+\kappa}|\lambda_2(t_i)\rangle+\sqrt{\tau_i}|\overline{\mu}_i\rangle+\tau_i^{\frac{1}{2}+\kappa}U(t_i)|\overline{\mu}_i\rangle \end{pmatrix} \right\rangle \\
&= \prod_{i\in I(\Theta)} 1+\tau_i\langle\overline{\nu}_i|\overline{\mu}_i\rangle+\tau_i^{1+\kappa}\left[c(t_i)+\langle\lambda_1(t_i)|\overline{\mu}_i\rangle+\langle\overline{\nu}_i|\lambda_2(t_i)\rangle+\langle\overline{\nu}_i|U(t_i)\overline{\mu}_i\rangle\right] \\
&= \prod_{i\in I(\Theta)} 1+\tau_i\langle\overline{\nu}_i|\overline{\mu}_i\rangle+\mathcal{O}(\tau_i^{1+\kappa}).
\end{aligned}$$

And since $\tau$ is the relevant order in this limit, as we have seen in equation (3.19), we have that

$$\varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \langle e_\Theta(\nu)|W_\Theta e_\Theta(\mu)\rangle = \varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \langle e_\Theta(\nu)|e_\Theta(\mu)\rangle\,.$$

Using the denseness of step functions we can lift this to proof to $L^2$ functions as outlined in figure 2.2. Hence $W_\Theta$ converges weak-* to the identity. □

One can see that the one in the $\mathbb{C}$-valued part is a necessary condition for this limit to be well defined. It is also noteworthy that changing the exponent from $\kappa$ to $-\kappa$ the resulting net in $\mathbb{C}$ wouldn't be convergent at all. Let us summarize this in a theorem.

**Theorem 5.3.** *The only possible form of an operator $W_\Theta \in \mathcal{B}(\mathcal{K}_\Theta)$ which converges in weak-* topology to a well defined object and does not contain trivial terms is given by*

$$W_\Theta = \bigotimes_{i\in I(\Theta)} \begin{pmatrix} 1+\tau_i\bar{c}_i & \sqrt{\tau_i}\,\langle\bar{\lambda}_i| \\ \sqrt{\tau_i}\,|\bar{\xi}_i\rangle & \mathbb{1}+\bar{U}_i \end{pmatrix} \in \mathcal{B}(\mathcal{K}_\Theta). \tag{5.6}$$

*With functions $c \in L^1([0,T],\mathcal{K})$, $|\lambda\rangle, |\xi\rangle \in L^2([0,T],\mathcal{K})$ and $U \in L^1([0,T],\mathcal{K})$.*

*Proof.* In analogy to the prior proof we calculate

$$\begin{aligned}
&\varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \langle e_\Theta(\nu)|W_\Theta e_\Theta(\mu)\rangle \\
&= \varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \prod_{i\in I(\Theta)} 1+\tau_i\left[\langle\bar{\nu}_i|\bar{\mu}_i\rangle + \bar{c}_i + \langle\bar{\lambda}_i|\bar{\mu}_i\rangle + \langle\bar{\nu}_i|\bar{\xi}_i\rangle + \langle\bar{\nu}_i|\bar{U}_i\bar{\mu}_i\rangle\right] \\
&\overset{(3.18)}{=} \exp\left(\int_0^T \langle\bar{\nu}_i|\bar{\mu}_i\rangle + \bar{c}_i + \langle\bar{\lambda}_i|\bar{\mu}_i\rangle + \langle\bar{\nu}_i|\bar{\xi}_i\rangle + \langle\bar{\nu}_i|\bar{U}_i\bar{\mu}_i\rangle\,\mathrm{d}t\right),
\end{aligned}$$

which is a non vanishing and non trivial expression in the limit. In the last equation we viewed $c, |\lambda\rangle, |\xi\rangle$ and $U$ as step functions via their averaged analogues $\bar{c}_i$ and so on. The proof lifts to $L^p$ functions, as outlined before.

Hence we see that only those $\tau$-orders contribute, especially the constant terms which make $W_\Theta$ the identity to zeroth-order in $\tau$. □

Having this corollary we know the relevant order of $\tau$ in each matrix element of our operators. We will use this to write down field operators up to leading orders in $\tau$, which will contribute in the inductive limit.

## 5.2 Constructing Weyl and Field Operators

We've already studied the importance of Weyl operators and their representation. Applying the worked out formalism of continuous measurements on Weyl operators seems to be a fruitful idea. We first define discrete Weyl operators on each time interval, starting with unitary rotations of the dilation spaces with respect to each other.

**Definition 5.4.** *Let $U : [0,T] \longrightarrow \mathcal{B}(\mathcal{K})$ be a unitary operator valued function. For a given $\Theta \in \mathfrak{Z}([0,T])$ we set:*

$$U_i \in \mathcal{B}(\mathcal{K}) \qquad U_i := U(t_i) \tag{5.7}$$

$$U_\Theta \in \mathcal{B}(\mathcal{K}_\Theta) \qquad U_\Theta := \bigotimes_{i \in I(\Theta)} \begin{pmatrix} 1 & 0 \\ 0 & U_i \end{pmatrix} \tag{5.8}$$

Before we add more to freedom to those operators we need to show their well-definedness in the inductive limit.

**Theorem 5.5.** *Let $U_\Theta \in \mathcal{B}(\mathcal{K}_\Theta)$ be as above . Then, for every $\Theta \in \mathfrak{Z}([0,T])$, the net $\Theta \longmapsto J_\Theta U_\Theta J_\Theta^\dagger \in \mathcal{B}\left(\mathcal{K}_{[0,T]}\right)$ converges in the strong topology to a unitary operator, i.e. for any convergent net $\Theta \longmapsto |\varphi_\Theta\rangle \in \mathcal{K}_\Theta$ and for all $\Theta \subseteq \Xi \subseteq \Lambda \in \mathfrak{Z}([0,T])$ we have that*

$$\lim_{\Lambda \gg \Xi} \|U_\Lambda J_{\Lambda\Theta}\varphi_\Theta - J_{\Lambda\Xi} U_\Xi J_{\Xi\Theta}\varphi_\Theta\| = 0. \tag{5.9}$$

*Proof.* Since $U(t)$ is unitary, every $U_\Theta$ is unitary and hence bounded. As seen in corollary 2.18, proving strong convergence reduces to showing that the following norm difference vanishes in the $\lim\limits_{\Lambda \gg \Xi}$-limit for all $\Theta \in \mathfrak{Z}([0,T])$ and $|\varphi_\Theta\rangle \in \mathcal{K}_\Theta$:

$$\begin{aligned}\|U_\Lambda J_{\Lambda\Theta}\varphi_\Theta - J_{\Lambda\Xi}U_\Xi J_{\Xi\Theta}\varphi_\Theta\|^2 &= \|U_\Lambda J_{\Lambda\Theta}\varphi_\Theta\|^2 + \|J_{\Lambda\Xi}U_\Xi J_{\Xi\Theta}\varphi_\Theta\|^2 \\ &\quad - 2\,\mathrm{Re}\,\langle U_\Lambda J_{\Lambda\Theta}\varphi_\Theta | J_{\Lambda\Xi}U_\Xi J_{\Xi\Theta}\varphi_\Theta\rangle \\ &= 2\|\varphi_\Theta\|^2 - 2\,\mathrm{Re}\left\langle J_{\Lambda\Xi}J_{\Xi\Theta}\varphi_\Theta \middle| U_\Lambda^\dagger J_{\Lambda\Xi}U_\Xi J_{\Xi\Theta}\varphi_\Theta\right\rangle \\ &= 2\|\varphi_\Theta\|^2 - 2\,\mathrm{Re}\left\langle J_{\Xi\Theta}\varphi_\Theta \middle| \left(J_{\Lambda\Xi}^\dagger U_\Lambda^\dagger J_{\Lambda\Xi}U_\Xi\right) J_{\Xi\Theta}\varphi_\Theta\right\rangle .\end{aligned}$$

Here we have used, that the $J_{\Xi\Theta}$ and the $U_\Theta$ are isometries for all $\Theta, \Xi \in \mathfrak{Z}([0,T])$. So the theorem is equivalent to the weak-* convergence of the map $J_{\Lambda\Xi}^\dagger U_\Lambda^\dagger J_{\Lambda\Xi} U_\Xi$ to the identity. Exploiting the explicit form of $J_{\Lambda\Xi}^\dagger U_\Lambda^\dagger J_{\Lambda\Xi}$ using lemma 5.1 and perform the matrix multiplication with $U_\Xi$ we have

$$J_{\Lambda\Xi}^\dagger U_\Lambda^\dagger J_{\Lambda\Xi} U_\Xi = \bigotimes_{i\in I(\Xi)} \begin{pmatrix} 1 & 0 \\ 0 & \sum\limits_{j\in I(\Lambda|_i)} \frac{\tau_j}{\tau_i} U_j^\dagger U_i \end{pmatrix} .$$

We are now restricting our unitary valued function $U \in L^1([0,T], \mathcal{B}(\mathcal{K}))$ to the dense subspace of step functions w.r.t. the coarsest interval decomposition of our analysis, i.e.

$$U(t) = \sum_{i\in I(\Xi)} U(t_i)\,\chi_{[t_{i-1},t_i)} = \sum_{i\in I(\Xi)} U_i\,\chi_i \in P_\Xi(L^1([0,T],\mathcal{B}(\mathcal{K}))).$$

Since $\Xi$ is coarser than $\Lambda$ it holds that for any $i \in I(\Xi)$ we have $U_i = U_j \ \forall j \in I(\Lambda|_i)$ and therefore, calculating the action of $J_{\Lambda\Xi}^\dagger U_\Lambda^\dagger J_{\Lambda\Xi} U_\Xi$ on an arbitrary element $|\varphi_\Xi\rangle := \bigotimes_{i\in I(\Xi)} (|a_i\rangle \quad |\alpha_i\rangle)^T \in \mathcal{K}_\Xi$ with $|a_i\rangle \in \mathbb{C}$ and $|\alpha_i\rangle \in \mathcal{K}$ we have:

$$\begin{aligned}\left\langle \varphi_\Xi \middle| J_{\Lambda\Xi}^\dagger U_\Lambda^\dagger J_{\Lambda\Xi} U_\Xi \varphi_\Xi \right\rangle &= \prod_{i\in I(\Xi)} \left\langle \begin{pmatrix} |a_i\rangle \\ |\alpha_i\rangle \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 \\ 0 & \sum\limits_{j\in I(\Lambda|_i)} \frac{\tau_j}{\tau_i} U_j^\dagger U_i \end{pmatrix} \begin{pmatrix} |a_i\rangle \\ |\alpha_i\rangle \end{pmatrix} \right\rangle \\ &= \prod_{i\in I(\Xi)} \left[ \langle a_l | a_l \rangle + \sum_{j\in I(\Lambda|_i)} \frac{\tau_j}{\tau_i} \left\langle \alpha_l \middle| U_j^\dagger U_i \alpha_l \right\rangle \right]\end{aligned}$$

$$= \prod_{i\in I(\Xi)} \left[ \langle a_l|a_l\rangle + \sum_{j\in I(\Lambda|_i)} \frac{\tau_j}{\tau_i} \left\langle \alpha_l \middle| U_i^\dagger U_i \alpha_l \right\rangle \right]$$
$$= \prod_{i\in I(\Xi)} \left[\langle a_l|a_l\rangle + \langle \alpha_l|\alpha_l\rangle\right] = \langle \varphi_\Xi|\varphi_\Xi\rangle\,.$$

This proves the weak-* convergence to the identity of $J^\dagger_{\Lambda\Xi} U^\dagger_\Lambda J_{\Lambda\Xi} U_\Xi$ and therefore the theorem for $U(t)$ being a step function. Via denseness, as discussed earlier in figure 2.2, this statement is also true for unitary valued $U \in L^1([0,T], \mathcal{B}(\mathcal{K}))$. □

Having this we can construct Weyl operators upon quantum fields. We start with a time independent construction and then add time dependence via $L^2$-functions, corresponding to our gauge freedom.

**Definition 5.6.** *Let $|\lambda\rangle \in \mathcal{K}$. We define the following* ***discrete field operator***

$$\Phi(\sqrt{\tau}\,|\lambda\rangle) \in \mathcal{B}(\mathbb{C}\oplus\mathcal{K}), \qquad \Phi(\sqrt{\tau}\,|\lambda\rangle) := \begin{pmatrix} 0 & \sqrt{\tau}\,\langle\lambda| \\ -\sqrt{\tau}\,|\lambda\rangle & 0 \end{pmatrix}. \tag{5.10}$$

*If we now add some time dependence, let $|\lambda\rangle \in L^2([0,T],\mathcal{K})$ and $\Theta \in \mathfrak{Z}([0,T])$. For $i \in I(\Theta)$ we write, as usual, $\left|\overline{\lambda}_i\right\rangle$ for the integral of $|\lambda(t)\rangle$ over the $[t_{i-1}, t_i)$ interval and $\tau_i = t_i - t_{i-1}$ for $t_i \in [0,T]$. We define the* ***discrete Weyl operator*** *to be*

$$W_\Theta(|\lambda\rangle) := \bigotimes_{i\in I(\Theta)} \exp\left(\Phi\left(\sqrt{\tau_i}\,\left|\overline{\lambda}_i\right\rangle\right)\right). \tag{5.11}$$

If we write $\left|\overline{\lambda}_i\right\rangle = \left\|\overline{\lambda}_i\right\|\,|e_i\rangle$ with $|e_i\rangle$ of unit length and $\theta_i = \sqrt{\tau_i}\left\|\overline{\lambda}_i\right\|$, one can explicitly calculate the operator exponential in multiple ways as

$$W_\Theta(|\lambda\rangle) = \bigotimes_{i\in I(\Theta)} \mathbb{1} + \Phi(\sqrt{\tau_i}\,\left|\overline{\lambda}_i\right\rangle) - \frac{\tau}{2}\begin{pmatrix} \left\langle\overline{\lambda}_i\middle|\overline{\lambda}_i\right\rangle & 0 \\ 0 & \left|\overline{\lambda}_i\right\rangle\!\left\langle\overline{\lambda}_i\right| \end{pmatrix} + \mathcal{O}(\tau^{3/2}) \tag{5.12}$$
$$= \bigotimes_{i\in I(\Theta)} \begin{pmatrix} \cos(\theta_i) & \sin(\theta_i\,\langle e_i|) \\ -\sin(\theta_i\,|e_i\rangle) & \cos(\theta_i\,|e_i\rangle\!\langle e_i|) \end{pmatrix} \tag{5.13}$$

$$= \bigotimes_{i \in I(\Theta)} \begin{pmatrix} \cos(\theta_i) & \sin(\theta_i) \langle e_i| \\ -\sin(\theta_i) |e_i\rangle & \mathbb{1}_{\mathcal{K}} + (\cos(\theta_i) - 1) |e_i\rangle\langle e_i| \end{pmatrix} \tag{5.14}$$

$$\overset{\text{t.r.o.}}{=} \bigotimes_{i \in I(\Theta)} \begin{pmatrix} 1 - \frac{\tau_i}{2} \langle \overline{\lambda}_i | \overline{\lambda}_i \rangle & \sqrt{\tau_i} \langle \overline{\lambda}_i | \\ -\sqrt{\tau_i} | \overline{\lambda}_i \rangle & \mathbb{1}_{\mathcal{K}} \end{pmatrix}, \tag{5.15}$$

where, here and from now on, "t.r.o." means "to relevant order in $\tau$". We used the sine and cosine functions which are $\mathcal{K}$, or even $\mathcal{B}(\mathcal{K})$, valued. Even though it should be implicitly clear what we mean by this, let us define those objects in the cosine case rigorously once. Potential misconceptions about the relation between the equations (5.13) and (5.14) should then be cleared.

**Notation 5.7.** *Let $\theta \in \mathbb{R}$ and $|e\rangle \in \mathcal{K}$. We define*

$$\cos(\theta\,|e\rangle) = \sum_{n=0}^{\infty} (-1)^n \frac{(\theta\,|e\rangle)^{2n}}{(2n)!}, \quad \cos(\theta\,|e\rangle\langle e|) = \sum_{n=0}^{\infty} (-1)^n \frac{(\theta\,|e\rangle\langle e|)^{2n}}{(2n)!}, \tag{5.16}$$

*where we set*

$$|e\rangle^{2n+1} := \underbrace{\langle e|e\rangle \cdots \langle e|e\rangle}_{n\text{-times}} |e\rangle\,, \qquad |e\rangle^0 := 1,$$
$$|e\rangle\langle e|^{2n+1} := |e\rangle \underbrace{\langle e|e\rangle \cdots \langle e|e\rangle}_{2n\text{-times}} \langle e|\,, \qquad |e\rangle\langle e|^0 := \mathbb{1}_{\mathcal{K}}.$$

*Keeping the parenthesis in mind and carefully rewriting the series above, one easily obtains that*

$$\sin(\theta_i)\langle e_i| = \sin(\theta_i \langle e_i|), \qquad \sin(\theta_i)|e_i\rangle = \sin(\theta_i |e_i\rangle)$$
$$\cos(\theta_i |e_i\rangle\langle e_i|) = \mathbb{1}_{\mathcal{K}} + (\cos(\theta_i) - 1)|e_i\rangle\langle e_i|\,.$$

Those equations will be used quite often, so one has to accurately take account of the parentheses. Now having a net of (bounded) operators, we need to show its convergence properties. We show that this net converges strongly to a well defined object.

**Theorem 5.8.** *Let* $|\lambda\rangle \in L^2([0,T],\mathcal{K})$ *and* $W_\Theta(|\lambda\rangle)$ *as in equation* (5.11)*, then the net* $\Theta \longmapsto J_\Theta W_\Theta(|\lambda\rangle)J_\Theta^\dagger$ *converges to a unitary operator in the strong topology, i.e. for any convergent net* $\Theta \longmapsto |\varphi_\Theta\rangle \in \mathcal{K}_\Theta$ *and for all* $\Theta \subseteq \Xi \subseteq \Lambda \in \mathfrak{Z}([0,T])$ *we have that*

$$\lim_{\Lambda \gg \Xi} \| W_\Lambda(|\lambda\rangle) J_{\Lambda\Theta} |\varphi_\Theta\rangle - J_{\Lambda\Xi} W_\Xi(|\lambda\rangle) J_{\Xi\Theta} |\varphi_\Theta\rangle \| = 0. \tag{5.17}$$

*Proof.* The proof is shifted to appendix A.3. □

Having shown convergence we are now able to define the limit operators in the following form.

**Definition 5.9.** *Let* $U \in L^1([0,T],\mathcal{B}(\mathcal{K}))$ *be an unitary operator valued function continuous in the strong topology and let* $|\lambda\rangle \in L^2([0,T],\mathcal{K})$*. We define the* ***continuous Weyl operators*** *to be:*

$$W(|\lambda\rangle) \in \mathcal{B}\left(\mathcal{K}_{[0,T]}\right) \qquad W(|\lambda\rangle) := \varinjlim_{\Theta \in \mathfrak{Z}([0,T])} W_\Theta(|\lambda\rangle) \tag{5.18}$$

$$W(|\lambda\rangle, U) \in \mathcal{B}\left(\mathcal{K}_{[0,T]}\right) \qquad W(|\lambda\rangle, U) := W(|\lambda\rangle) \varinjlim_{\Theta \in \mathfrak{Z}([0,T])} U_\Theta. \tag{5.19}$$

We have to justify the name of these operators and show that they obey the Weyl CCR, as analyzed before. The Weyl CCR can be seen as a projective and unitary representation of a symplectic space. In our case we have the Hilbert space $\mathcal{K}_{[0,T]}$ which induces a symplectic form, using its complex structure, via the imaginary part.

The Weyl operators defined above, however, inherit a unitary "intertwiner" which makes the representation a bit more complicated. Let us analyze this more careful using the notion of semi direct products.

**Proposition 5.10.** *The Hilbert space* $\mathcal{K}_{[0,T]}$ *canonically induces the two groups*

$$G_1 = \left(L^2\left([0,T],\mathcal{K}_{[0,T]}\right), +\right), \qquad G_2 = \left(L^1\left([0,T], U\left(\mathcal{K}_{[0,T]}\right)\right), \circ\right) \tag{5.20}$$

*with* $U\left(\mathcal{K}_{[0,T]}\right)$ *denoting the unitary operators on* $\mathcal{K}_{[0,T]}$ *equipped with concatenation. The homomorphism* $\theta$ *given by*

$$\theta : G_2 \longrightarrow Aut(G_1), \qquad \theta(U)(|\mu\rangle) := U\,|\mu\rangle\,, \tag{5.21}$$

*defines a semi direct product w.r.t. $\theta$, denoted by $(G_1 \rtimes_\theta G_2, \bullet)$ as a subset of the Cartesian product $G_1 \times G_2$ via*

$$\bullet : (G_1 \rtimes_\theta G_2) \times (G_1 \rtimes_\theta G_2) \longrightarrow (G_1 \rtimes_\theta G_2)$$
$$(|\lambda\rangle\,, U_1) \bullet (|\mu\rangle\,, U_2) := (|\lambda\rangle + \theta(U_1)(|\mu\rangle), U_1 \circ U_2) = (|\lambda\rangle + U_1\,|\mu\rangle\,, U_1 U_2). \tag{5.22}$$

*A unitary projective representation of $(G_1 \rtimes_\theta G_2, \bullet)$ on $\mathcal{K}_{[0,T]}$ would then be given by a homomorphism $W : (G_1 \rtimes_\theta G_2, \bullet) \longrightarrow \left(U\left(\mathcal{K}_{[0,T]}\right), \circ\right)$ satisfying*

$$W(|\lambda\rangle\,, U_1) \circ W(|\mu\rangle\,, U_2) = c((|\lambda\rangle\,, U_1), (|\mu\rangle\,, U_2))\, W((|\lambda\rangle\,, U_1) \bullet (|\mu\rangle\,, U_2)), \tag{5.23}$$

*with $c \in \mathbb{C}$ being of unit length.*

It should be obvious, that the outlined maps truly are homomorphisms and that the construction is therefore well defined. We will omit the notational overload coming from all those different group structures and simply refer to this group representation above that $W$ is intertwined by a unitary action. That the Weyl operators we encountered are of this form is the statement of the following theorem.

**Theorem 5.11.** *The operator $W(|\lambda\rangle\,, U)$ defined as above with $|\lambda\rangle \in L^2([0,T], \mathcal{K})$ fulfills the Weyl commutation relations on the representation space $\mathcal{K}_{[0,T]}$ with a unitary intertwiner.*

*That is for $|\lambda\rangle\,, |\mu\rangle \in L^2([0,T], \mathcal{K})$ and $U_1, U_2 \in L^1\left([0,T], U\left(\mathcal{K}_{[0,T]}\right)\right)$ unitary operator valued functions, continuous in the strong topology, we have that $W$ is a infinite-dimensional, projective and unitary representation of the Weyl CCR with*

$$W(|\lambda\rangle\,, U_1) W(|\mu\rangle\,, U_2) = e^{-i\,T\,\mathrm{Im}(\langle\lambda|U_1\mu\rangle)}\, W(|\lambda\rangle + U_1\,|\mu\rangle\,, U_1 U_2). \tag{5.24}$$

*Proof.* Note that equation (5.24) is a projective representation like in equation (5.23) with symplectic form $\sigma = -2T\,\mathrm{Im}(\langle\lambda|\mu\rangle)$. Since all operators involved are bounded, they commute with the strong limits.
We split the proof in various steps.

**Step 1.**

We start be proving the statement with $U_1 = U_2 = \mathbb{1}$, i.e.

$$\varinjlim_{\Theta \in \mathfrak{Z}([0,T])} W_\Theta(|\lambda\rangle) W_\Theta(|\mu\rangle) W_\Theta(-(|\lambda\rangle + |\mu\rangle))\, e^{i\,T\,\mathrm{Im}(\langle\lambda|\mu\rangle)} = \mathbb{1}.$$

An explicit calculation to relevant order in $\tau$ of $W_\Theta(|\lambda\rangle) W_\Theta(|\mu\rangle) W_\Theta(-(|\lambda\rangle + |\mu\rangle))$ can be found in appendix A.4. One sees that

$$W_\Theta(|\lambda\rangle) W_\Theta(|\mu\rangle) W_\Theta(-(|\lambda\rangle + |\mu\rangle)) \overset{\text{t.r.o.}}{=} \bigotimes_{i \in I(\Theta)} \begin{pmatrix} 1 - i\tau_i \,\mathrm{Im}\left(\langle \overline{\lambda}_i | \overline{\mu}_i \rangle\right) & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

We need to analyze to which operator $W$ this concatenated Weyl operator converges to. As seen before we need to analyze how this operator interacts with the embeddings $J_{\Xi\Theta}$ as seen in theorem 5.1 and use the limit obtained in equation (3.18) for the resulting factor, i.e.

$$\begin{aligned} &\varinjlim_{\Theta \in \mathfrak{Z}([0,T])} J^\dagger_{\Xi\Theta} W_\Xi(|\lambda\rangle) W_\Xi(|\mu\rangle) W_\Xi(-(|\lambda\rangle + |\mu\rangle)) J_{\Xi\Theta} \\ &\quad = \varinjlim_{\Theta \in \mathfrak{Z}([0,T])} \bigotimes_{i \in I(\Theta)} \prod_{j \in I(\Xi|_i)} \left(1 - i\tau_j \,\mathrm{Im}\left(\langle \overline{\lambda}_j | \overline{\mu}_j \rangle\right)\right) \mathbb{1}_{\mathbb{C}\oplus\mathcal{K}} \\ &\quad = \exp\left( - \int_0^T i \,\mathrm{Im}\left(\langle \overline{\lambda} | \overline{\mu} \rangle\right) \right) = \exp\bigl(-i\,T\,\mathrm{Im}\left(\langle \overline{\lambda} | \overline{\mu} \rangle\right)\bigr). \end{aligned}$$

Hence, by multiplying with $\exp\bigl(i\,T\,\mathrm{Im}\left(\langle \overline{\lambda} | \overline{\mu} \rangle\right)\bigr)$, we get the desired formula above.

**Step 2.**

We fix one unitary valued function $U$ and show

$$W(0, U^\dagger) W(|\lambda\rangle) W(0, U) = W(U^\dagger\, |\lambda\rangle).$$

First observe that terms like $W_\Theta(0) = \exp(0) = \mathbb{1}$ are neglected. One can calculate

$$\begin{aligned}
W(0,U^\dagger)W(|\lambda\rangle)W(0,U) &= \varinjlim_{\Theta\in\mathfrak{Z}([0,T])} U_\Theta^\dagger\, W_\Theta(|\lambda\rangle)\, U_\Theta \\
&= \varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \bigotimes_{i\in I(\Theta)} \begin{pmatrix} 1 & 0 \\ 0 & U_i^\dagger \end{pmatrix} \exp\left(\Phi\left(\sqrt{\tau_i}\,|\overline{\lambda}_i\rangle\right)\right) \begin{pmatrix} 1 & 0 \\ 0 & U_i \end{pmatrix} \\
&= \varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \bigotimes_{i\in I(\Theta)} \begin{pmatrix} \cos(\theta_i) & \sin(\theta_i)\,\langle e_i|\,U_i \\ -\sin(\theta_i)\,U_i^\dagger\,|e_i\rangle & U_i^\dagger\cos(\theta_i\,|e_i\rangle\!\langle e_i|)U_i \end{pmatrix} \\
&= \varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \bigotimes_{i\in I(\Theta)} \exp\left(\Phi\left(\sqrt{\tau_i}\,U_i^\dagger\,|\overline{\lambda}_i\rangle\right)\right) = W(U^\dagger\,|\lambda\rangle),
\end{aligned}$$

where we used the short notation $\theta_i = \sqrt{\tau_i}\,\left\|\overline{\lambda}_i\right\|$.

**Step 3.**

There is one observation left to be done and this is rather trivial. One can see easily that

$$W(0,U_1)W(0,U_2) = W(0,U_1U_2).$$

**Step 4.**

Combining everything we can calculate

$$\begin{aligned}
W(|\lambda\rangle\,,U_1)W(|\mu\rangle\,,U_2) &\overset{\text{Def.}}{=} W(|\lambda\rangle)W(0,U_1)W(|\mu\rangle)W(0,U_2) \\
&\overset{\text{Step 3.}}{=} W(|\lambda\rangle)W(0,U_1)W(|\mu\rangle)W(0,U_1^\dagger)W(0,U_1)W(0,U_2) \\
&\overset{\text{Step 2./3.}}{=} W(|\lambda\rangle)W(U_1\,|\mu\rangle)W(0,U_1U_2) \\
&\overset{\text{Step 0./1.}}{=} e^{-i\,T\,\mathrm{Im}(\langle\lambda|U_1\mu\rangle)}\,W(|\lambda\rangle + U_1\,|\mu\rangle)W(0,U_1U_2) \\
&\overset{\text{Def.}}{=} e^{-i\,T\,\mathrm{Im}(\langle\lambda|U_1\mu\rangle)}\,W(|\lambda\rangle + U_1\,|\mu\rangle\,,U_1U_2).
\end{aligned}$$

So the theorem is proven. □

# Chapter 6

# Point Processes and Counting Statistics

When we consider a POVM $A_i$ describing a measurement in the i-th time-step and ignoring all other time-steps, then it would certainly have the form $(A_i(x)@i)$ and $x \longmapsto A_i(x)$ is a POVM in $\mathcal{B}(\mathbb{C} \oplus \mathcal{K})$, where we have used the notation in equations (3.24) and (3.25). Summing over all subintervals would then capture the whole observable algebra $\mathcal{B}(\mathcal{K}_\Theta)$.

This would correspond to an operator second quantization and it seems to be a natural way to model measurements in modern laboratories. Outcomes of those measurements are then going to be describes by so-called point processes. Let us elaborate this idea a bit further.

## 6.1 Field Operators and Second Quantization

One stumbles upon second quantization mainly in the sake of many-particle quantum mechanics to define a "many-particle" Hilbert space, the Fock space, out of a given one-particle Hilbert space. Second quantization, in this thesis however, deals with operators on tensor product spaces, w.r.t. a certain interval decomposition.

Since the basic mathematical idea, i.e. lifting an element from one space to an object on (every possible) tensor product of this space with itself, we chose to name both notions identically.

**Definition 6.1.** *Let $\Theta \in \mathfrak{Z}([0,T])$. We define second quantized operators via the bosonic/fermionic functors $\Gamma_\Theta, \Gamma_{\Theta-} : \mathcal{B}(\mathbb{C} \oplus \mathcal{K}) \longrightarrow \mathcal{B}(\mathcal{K}_\Theta)$, defined as follows.*

*Having an operator valued function $A : [0,T] \longrightarrow \mathcal{B}(\mathbb{C} \oplus \mathcal{K})$ we set*

$$\Gamma_\Theta(A(t)) := \sum_{i\in I(\Theta)} (A(t_i)@i) \qquad \textit{and} \qquad \Gamma_{\Theta-}(A(t)) := \sum_{i\in I(\Theta)} (A(t_i)@i)_-, \tag{6.1}$$

*where we have used the "@i" notations from equation* (3.24) *and* (3.25) *respectively.*

Note that the operators could be independent of the time-step, i.e. $A(t_i) = A$ for all $i \in I(\Theta)$. It therefore makes sense for $\Gamma_\Theta, \Gamma_{\Theta-}$ to act on constant operators as well. It should always be clear from context what we mean in the particular case.

We will restrict our analysis to bosonic fields, as outlined before. An important feature of the bosonic functor, that we will use in some proofs, is explained in the following corollary.

**Corollary 6.2.** *Let $A, B$ be operator functions in $\mathcal{B}(\mathbb{C}\oplus\mathcal{K})$ like above. It holds that the bosonic field operators a compatible with the commutator, i.e.*

$$[\Gamma_\Theta(A), \Gamma_\Theta(B)] = \Gamma_\Theta([A, B]). \tag{6.2}$$

*The functor $\Gamma_\Theta$ is therefore a Lie algebra homomorphism between the Lie algebras $(\mathcal{B}(\mathbb{C}\oplus\mathcal{K}), [\cdot,\cdot])$ and $(\mathcal{B}(\mathcal{K}_\Theta), [\cdot,\cdot])$.*

*Proof.* Additivity and $\mathbb{C}$-linearity is trivial. Inserting the definitions and using that the tensor product of linear maps acts factor-wise one easily calculates the only non vanishing terms in the bosonic part are the factors in which $A$ and $B$ act together.

This is because every term with an $A$ acting at the i-th and a $B$ acting at the j-th site always cancels with a corresponding term with the opposite sign. The $AB$ terms do not cancel, since they are subtracted with corresponding $BA$ terms, which proofs the corollary. □

Since the inductive limit space $\mathcal{K}_{[0,T]}$ is isomorphic to the (for us bosonic) Fock space, one would be interested in building "true" Fock space functors. In the measurement setup this would correspond to an embedding from $\mathbb{C}\oplus\mathcal{K}$ into $\mathcal{K}_{[0,T]}$, defined on the discrete subspaces. This motivates the following approach.

**Definition 6.3.** *Let $\Theta \in \mathfrak{Z}([0,T])$. The bosonic Fock space functor $\Gamma$ is in the context of continuous measurement given by the natural embedding w.r.t. the given interval decomposition, i.e.*

$$\Gamma : \mathcal{B}(\mathbb{C}\oplus\mathcal{K}) \longrightarrow \mathcal{B}\left(\mathcal{K}_{[0,T]}\right),$$

$$A \longmapsto \Gamma(A) := \varinjlim_{\Theta \in \mathfrak{Z}([0,T])} J_\Theta \Gamma_\Theta(A) J_\Theta^\dagger. \tag{6.3}$$

It seems to be a natural idea to see what the second quantized Fock space analogues of the discrete field operators look like. Indeed, having a Fock space functor simplifies a lot of tasks regarding Weyl operators as we shall see.

**Definition 6.4.** *Given a function $|\lambda\rangle \in L^2([0,T],\mathcal{K})$ and $\Theta \in \mathfrak{Z}([0,T])$. Recall the definition of the discrete field operators in equation (5.10). We define the* ***second quantized discrete (bosonic) field operator*** *to be*

$$\begin{aligned} \Phi_\Theta : L^2([0,T],\mathcal{K}) &\longrightarrow \mathcal{B}(\mathcal{K}_\Theta) \\ |\lambda\rangle &\longmapsto \Phi_\Theta(|\lambda\rangle) := \Gamma_\Theta\left(\Phi\left(\sqrt{\tau_i}\,\left|\overline{\lambda}_i\right\rangle\right)\right). \end{aligned} \tag{6.4}$$

Since we defined those field operators on a discrete Hilbert space, i.e. on $\mathcal{K}_\Theta$, it is a priori not clear that the CCR is truly fulfilled and that $\Phi_\Theta$ therefore deserves this name. This observation is the statement of the following theorem.

**Theorem 6.5.** *The field operators obey the CCR in the inductive limit, i.e. for $|\lambda\rangle, |\mu\rangle \in L^2([0,T],\mathcal{K})$ we have*

$$\varinjlim_{\Theta \in \mathfrak{Z}([0,T])} [\Phi_\Theta(|\lambda\rangle), \Phi_\Theta(|\mu\rangle)] = -i\sigma_{anti}(|\lambda\rangle, |\mu\rangle)\,\mathbb{1}_{\mathcal{K}_{[0,T]}}, \tag{6.5}$$

*with a nondegenerate bilinear anti-symmetric form $\sigma_{anti}(x,y) = 2T\,\mathrm{Im}\left(\langle x|y\rangle\right)$.*

*Proof.* We start by explicitly calculating the commutator in a discrete setting.

$$\begin{aligned} &[\Phi_\Theta(|\lambda\rangle), \Phi_\Theta(|\mu\rangle)] = \Phi_\Theta(|\lambda\rangle)\,\Phi_\Theta(|\mu\rangle) - \Phi_\Theta(|\mu\rangle)\,\Phi_\Theta(|\lambda\rangle) \\ &\quad = \Gamma_\Theta\left(\Phi\left(\sqrt{\tau_i}\,\left|\overline{\lambda}_i\right\rangle\right)\right)\,\Gamma_\Theta\left(\Phi\left(\sqrt{\tau_i}\,\left|\overline{\mu}_i\right\rangle\right)\right) - \Gamma_\Theta\left(\Phi\left(\sqrt{\tau_i}\,\left|\overline{\mu}_i\right\rangle\right)\right)\,\Gamma_\Theta\left(\Phi\left(\sqrt{\tau_i}\,\left|\overline{\lambda}_i\right\rangle\right)\right) \\ &\quad \overset{(6.2)}{=} \Gamma_\Theta\left(\Phi\left(\sqrt{\tau_i}\,\left|\overline{\lambda}_i\right\rangle\right)\,\Phi\left(\sqrt{\tau_i}\,\left|\overline{\mu}_i\right\rangle\right) - \Phi\left(\sqrt{\tau_i}\,\left|\overline{\mu}_i\right\rangle\right)\,\Phi\left(\sqrt{\tau_i}\,\left|\overline{\lambda}_i\right\rangle\right)\right) \\ &\quad = \Gamma_\Theta\left(\tau_i \begin{pmatrix} 0 & \left\langle\overline{\lambda}_i\right| \\ -\left|\overline{\lambda}_i\right\rangle & 0 \end{pmatrix} \begin{pmatrix} 0 & \left\langle\overline{\mu}_i\right| \\ -\left|\overline{\mu}_i\right\rangle & 0 \end{pmatrix} - \tau_i \begin{pmatrix} 0 & \left\langle\overline{\mu}_i\right| \\ -\left|\overline{\mu}_i\right\rangle & 0 \end{pmatrix} \begin{pmatrix} 0 & \left\langle\overline{\lambda}_i\right| \\ -\left|\overline{\lambda}_i\right\rangle & 0 \end{pmatrix}\right) \end{aligned}$$

$$= \Gamma_\Theta \left( \tau_i \begin{pmatrix} -\langle\overline{\lambda}_i|\overline{\mu}_i\rangle & 0 \\ 0 & -|\overline{\lambda}_i\rangle\langle\overline{\mu}_i| \end{pmatrix} - \begin{pmatrix} -\langle\overline{\mu}_i|\overline{\lambda}_i\rangle & 0 \\ 0 & -|\overline{\mu}_i\rangle\langle\overline{\lambda}_i| \end{pmatrix} \right)$$

$$= \Gamma_\Theta \left( \tau_i \begin{pmatrix} -2i\,\mathrm{Im}(\langle\overline{\lambda}_i|\overline{\mu}_i\rangle) & 0 \\ 0 & -2i\,\mathrm{Im}\,|\overline{\lambda}_i\rangle\langle\overline{\mu}_i| \end{pmatrix} \right)$$

$$= \Gamma_\Theta \left( -2i\,\mathrm{Im}(\langle\overline{\lambda}_i|\overline{\mu}_i\rangle)\tau_i\,\mathbb{1}_{\mathbb{C}\oplus\mathcal{K}} \right) = -i\sigma_{\mathrm{anti}}(|\lambda\rangle\,,|\mu\rangle)\mathbb{1}_{\mathcal{K}_\Theta}$$

The imaginary part, and therefore the complex conjugation in one of the last equations, is meant to be taken point-wise on the $\mathcal{B}(\mathcal{K})$-valued part. We also used the identity $\Gamma_\Theta\left(\tau_i\,\mathbb{1}_{\mathbb{C}\oplus\mathcal{K}}\right) = T\,\mathbb{1}_{\mathcal{K}_\Theta}$, which should be trivial to see, given the definition of $\Gamma_\Theta$. □

## 6.2 Counting Statistics in Stochastic Calculus

The main purpose of this work is to connect one-dimensional quantum fields and their counting statistics, to the subject of continuous measurements. Therefore we need to recapitulate some basic knowledge in stochastic calculus, the reader, however, should be familiar with $\sigma$-algebras and measures. The notation is based on [KRW13].

**Notation 6.6.** *Let $\mu$ be an operator or scalar valued measure. We abbreviate the integral over scalar functions $f$ as*

$$\mu\,[f] := \int \mu(\mathrm{d}x) f(x). \tag{6.6}$$

*Respectively for POVMs $\{F_x \,|\, x \in X\}$ one defines $F[f] := \sum_{x\in X} f(x)F_x$ in the case that $X$ is discrete and $F[f] = \int_X f(x)F(\mathrm{d}x)$ if $X$ is continuous.*

We now concentrate on so-called **point processes** which are special probability measures. Those are given by certain random measures with possible outcomes $(x(1), x(2), \dots)$ over a set $X$. Every point process that can be represented by a weighted sum[1] of Dirac measures, i.e.

[1] The Dirac measure therefore counts the number of occurrences of a given outcome including its multiplicities.

$$\xi = \sum_{i}^{n} \delta_{x(i)}, \tag{6.7}$$

where $x(i) \in X$ and $n$ is a random number, is called a **proper point process**. As outlined in [LP17][Chapter 2.4.] one knows that every point process is proper, as long as its **state space** $X$ is at least a Borel subspace of a complete separable metric space. We will always assume this to be the case.

Every Dirac peak can then be interpreted as a click-event. Since most quantum experiments (especially including photons) are more of a beam-like character and one doesn't measure one-particle observables, but rather a series of click events over time and/or space, one does not obtain a single probability distribution as output but something more like a distribution of outcomes.
Hence point processes seem like a natural description of physical processes in modern laboratories. Since the viewpoint of point processes being "some random points in some output space" is too vague, we introduce them more formally:

**Definition 6.7.** *Let $X$ be at least a complete separable metric space with Borel $\sigma$-algebra $\mathfrak{X}$.*

- *A **boundedly-finite** measure on $(X, \mathfrak{X})$ is a measure $\nu$ s.t. $\nu(E) < \infty$ for all bounded $E \in \mathfrak{X}$.*
- *A **counting measure** is a boundedly finite measure with $\nu(E) \in \mathbb{N}_0 \; \forall \, E \in \mathfrak{X}$. We denote the set of counting measures by $\mathcal{CM}(X, \mathfrak{X})$.*
- *A **random measure** over $(X, \mathfrak{X})$ is a measurable mapping from a probability space $(\Omega, \Sigma, \mu)$ into the boundedly-finite measures on $(X, \mathfrak{X})$.*
- *A **point process** is then just a counting measure valued random measure, i.e. a measurable map*

$$\xi : (\Omega, \Sigma, \mu) \longrightarrow \mathcal{CM}(X, \mathfrak{X}), \quad \textit{i.e.} \quad \xi(\omega, E) \in \mathbb{N}_0 \; \forall \, \omega \in \Omega, E \in \mathfrak{X}. \tag{6.8}$$

So a point process is a special case of a stochastic process. As usual for point processes (cf. [MS06][p. 267 et seqq.]), we will omit the argument coming from the probability space and view point processes synonymously via their image, i.e. a counting measure.

Before we define counting statistics we need two objects, which we will be helpful in the upcoming analysis. First we want to define some probability measures on a small domain and "lift" the definition to a bigger domain.

**Definition 6.8.** *Let $(X_1, \Sigma_1)$, $(X_2, \Sigma_2)$ be measurable spaces and $\Upsilon : X_1 \longrightarrow X_2$ a measurable map. Then, given a measure $\mu : \Sigma_1 \longrightarrow [0, \infty]$, one can define the* ***pushforward measure*** *on $(X_2, \Sigma_2)$ w.r.t. $\Upsilon$ as*

$$\Upsilon_*(\mu) : \Sigma_2 \longrightarrow [0, \infty], \qquad (\Upsilon_*(\mu))(B) := \mu\left(\Upsilon^{-1}(B)\right). \tag{6.9}$$

Another tool we need is to define a measure via its kernel.

**Definition 6.9.** *Let $(X_1, \Sigma_1)$, $(X_2, \Sigma_2)$ again be measurable spaces. A function*

$$\kappa : X_1 \times \Sigma_2 \longrightarrow [0, \infty] \tag{6.10}$$

*is called a* ***(transition) kernel*** *from $X_1$ to $X_2$ iff:*

*i) The map $x_1 \longmapsto \kappa(x_1, S_2)$ is measurable on $(X_1, \Sigma_1)$ for every fixed $S_2 \in \Sigma_2$.*

*ii) The map $S_2 \longmapsto \kappa(x_1, S_2)$ is a measure on $(X_2, \Sigma_2)$ for every fixed $x_1 \in X_1$.*

*If $\kappa$ maps into $[0, 1]$ and the corresponding measures are all probability measures, then $\kappa$ is called a* ***Markov kernel****.*

This definition shows us that a point process like in equation (6.8) can be defined equivalently as a transition kernel from $\Omega$ to $X$. We now have everything we need to talk about characteristic functions and their physical interpretation.

Instead of calculating probability measures, it is sometimes way more practical to look at its **characteristic function** which is given by its Fourier transform, i.e. the expectation value of the function $\xi \longmapsto \exp(i\xi[f])$, hence

$$C(f) = \left\langle e^{i\xi[f]} \right\rangle. \tag{6.11}$$

Depending on the expectation value, we require $f$ to be integrable, continuous,... s.t. the expression above is well defined.

Probability measures and characteristic functions stand in bijection to each other; thereby no information about the underlying stochastic process will be lost.
If a certain property of the point process is fulfilled, namely **complete randomness**, then one obtains a **Poisson process**

$$\mathrm{Poi} : (\Omega, \Sigma, \mu) \longrightarrow \mathcal{CM}(X, \mathfrak{X}), \quad \{n_1, \dots, n_k\} \longmapsto \mathrm{Poi}_n := \mathrm{Poi}(\{n_1, \dots, n_k\}, \cdot),$$

which is defined via the **intensity measure** $\mu$ on $(X, \mathfrak{X})$, s.t.

$$\mathrm{Poi}_n(\{X_1, \dots, X_k\}) = \prod_{l=1}^{k} \frac{\mu(X_l)^{n_l}}{n_l!} \exp(-\mu(X_l)). \tag{6.12}$$

The complete randomness condition ensures that the particle statistics in every bounded subset of $X_l \subseteq X$ are independent from each other, which seems to model particle beams accurately. Having this measure one can calculate its characteristic function straightforward to

$$C(f) = \exp\bigg(\int \mu(\mathrm{d}x)(e^{if} - 1)\bigg). \tag{6.13}$$

The $\mathbf{k^{th}}$ **moment** of the point process $\xi$ is the unique permutation symmetric measure $m_k$ on $X^k$ that satisfies

$$\int m_k(\mathrm{d}x_1, \dots, \mathrm{d}x_k) \prod_{j=1}^{k} f(x_j) = \Big\langle \xi[f]^k \Big\rangle. \tag{6.14}$$

For arbitrary functions $f$ one can simply write (see [KRW13])

$$m_k[f] = \Bigg\langle \sum_{i_1,\dots,i_k} f(x(i_1), \dots, x(i_k)) \Bigg\rangle. \tag{6.15}$$

The moments are physically very important, since the first and second moment correspond to the mean value and variance of the underlying point process. With the characteristic function, however, one is able to extract every possible (existing) moment, since one has

$$C(f) = \sum_k \frac{i^k}{k!} \int m_k(\mathrm{d}x_1 \cdots \mathrm{d}x_k) \prod_{j=1}^{k} f(x_j).$$

That means that one can extract the k-th moment from $C(\lambda f)$ by differentiating k times w.r.t. a generic $\lambda \in \mathbb{R}$ and evaluated at $\lambda = 0$. Since in some probability moments singularities tend to pop up, it is sometimes more convenient to define more regular objects, so-called "factorial moments", in the following way:

$$\widehat{m}_k[f] = \left\langle \sum_{\substack{i_1,\ldots,i_k \\ \text{distinct}}} f(x(i_1),\ldots,x(i_k)) \right\rangle .$$

Their generating function

$$\widehat{C}(f) = \sum_k \frac{1}{k!} \int \widehat{m}_k(\mathrm{d}x_1 \cdots \mathrm{d}x_k) \prod_{j=1}^{k} f(x_j), \tag{6.16}$$

is therefore related to the original characteristic function via $C(f) = \widehat{C}(e^{if} - 1)$.

The reason one takes counting statistics into account is that it makes the construction of second quantized observables on Fock space fairly easy [Wer89]. The idea is to take an observable $F$, given as a POVM, acting on a one-particle Hilbert space $\mathcal{H}$ and lift it to a second quantized observable $\Gamma F$ by measuring $F$ on all particles that arrive, restricted to the N-particle space. We will use this idea later on, talking about "measurements on arrival". Formalizing this idea we get

$$\left((\Gamma F)\left[e^{i\xi[f]}\right]\right)_N = \int F(\mathrm{d}x_1) \otimes \cdots \otimes F(\mathrm{d}x_N)\; e^{\sum_i f(x_i)} = \left(F\left[e^{if}\right]\right)^{\otimes N} .$$

To know how this operator acts on full Fock space one needs to take the direct sum over all $N \in \mathbb{N}$, i.e. one obtains

$$(\Gamma F)\left[e^{i\xi[f]}\right] = \Gamma\left(F\left[e^{if}\right]\right). \tag{6.17}$$

Hence, given a state $\rho$, one can extract the full counting statistics w.r.t. the second quantized POVM $\Gamma F$ from the characteristic function

$$C(f) = \mathrm{tr}\Big(\rho\,(\Gamma F)\left[e^{i\xi[f]}\right]\Big) \overset{(6.17)}{=} \mathrm{tr}\Big(\rho\,\Gamma\left(F\left[e^{if}\right]\right)\Big). \tag{6.18}$$

Using equation (6.16) with the correspondence between $C(f)$ and $\widehat{C}(f)$ combined with equation (6.17) we see

$$\widehat{C}(f) = \operatorname{tr}\left(\rho\,\Gamma\left(\mathbb{1} + F[f]\right)\right) = \sum_{N=0}^{\infty} \operatorname{tr}\left(\rho_N \left(\mathbb{1} + F[f]\right)^{\otimes N}\right) = \sum_{k=0}^{\infty} \operatorname{tr}\left(\hat{\rho}_k\, F[f]^{\otimes k}\right),$$

with the **reduced k-particle density operator**

$$\hat{\rho}_k = \sum_{N=k}^{\infty} \binom{N}{k} \operatorname{tr}_{[k+1,\dots,N]} \rho_N. \tag{6.19}$$

We will now use this correspondence between second quantized POVMs and their characteristic functions, by explicitly constructing a POVM of desire. We already defined the Fock space functor corresponding to continuous measurements in equation (6.3).

## 6.3 Measurement on Arrival POVMs

We have already seen how to second quantize observables in equation (6.17). However, if we want to analyze counting statistics, we need an extra notion of arrival times. These construction is going to be the task of the next definitions and theorems. Since we are not interested in the total number of events, but rather in their distribution in time, we need to modify our measure structure in the following way, starting with an abbreviation:

**Definition 6.10.** *Let $\Theta \in \mathfrak{Z}([0,T])$ and $U_i \in \mathcal{B}(\mathcal{K})$ for all $i \in I(\Theta)$ be a family of operators. In abuse of notation we define:*

$$A_i \in \mathcal{B}(\mathbb{C} \oplus \mathcal{K}) \qquad A_i := \begin{pmatrix} 0 & 0 \\ 0 & U_i \end{pmatrix}$$

$$\Gamma_\Theta(U_i) \in \mathcal{B}(\mathcal{K}_\Theta) \qquad \Gamma_\Theta(U_i) := \Gamma_\Theta(A_i). \tag{6.20}$$

If we want to describe a measurement on arrival, we need to define "what to measure" if no particles arrive in a certain time-step. Therefore we need to get a more advanced notion of a POVM by increasing its domain.

**Definition 6.11.** *Let $X$ be the non void set*[2] *of "events" with Borel $\sigma$-algebra $\mathfrak{X}$. We define that the label "0" corresponds to "no-event" and therefore demand $0 \notin X$. Furthermore let $\Theta \in \mathfrak{Z}([0,T])$ and for all $i \in I(\Theta)$ let $F_i : \mathfrak{X} \longrightarrow \mathcal{B}(\mathcal{K})$ be a POVM. A set $\xi \subseteq \{0\} \cup X$ is defined to be measurable iff $\xi \cap X$ is measurable in $X$.*

*We denote the (canonical) $\sigma$-algebra of the set $\{0\} \cup X =: X^0$ by $\mathfrak{X}^0$ and define a new POVM $M_i : \mathfrak{X}^0 \longrightarrow \mathcal{B}(\mathbb{C} \oplus \mathcal{K})$ via*

$$M_i(\{0\}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad M_i(\mathcal{X}) = \begin{pmatrix} 0 & 0 \\ 0 & F_i(\mathcal{X}) \end{pmatrix} \quad \text{for} \quad \mathcal{X} \in \mathfrak{X}. \tag{6.21}$$

*The measure of a subset of $\{0\} \cup X$ is then the sum of its measures in $\{0\}$ and $\{X\}$ and this makes $M_i$ clearly to a POVM, describing* ***measurements on arrival****.*

This POVM models the situation where in the time interval $[t_{i-1}, t_i)$ no particle arrives, described by $M_i(\{0\})$, or a particle arrives and we measure according the POVM $F$. The term "measurement on arrival" should become clear in this context.

## 6.4 Constructing Point Processes

Having this POVM we can construct a probability measure for our measurement on arrival and build a true point process upon it.

**Definition 6.12.** *In the situation from above we equip the Cartesian product state space w.r.t. the interval decomposition $\Theta$, i.e.*

$$X^0_\Theta := \prod_{i \in I(\Theta)} \{0\} \cup X, \tag{6.22}$$

*with its product Borel $\sigma$-algebra $\mathfrak{X}^0_\Theta$ and define a probability measure $\nu_\rho$ for a given state $\rho \in \mathfrak{T}(\mathcal{K})$ and $\xi_i \in \mathfrak{X}^0$ for $i \in I(\Theta)$ as*

$$\nu_\rho : \mathfrak{X}^0_\Theta \longrightarrow [0,1], \qquad \prod_{i \in I(\Theta)} \xi_i \longmapsto \operatorname{tr}\left( J^\dagger_\Theta \rho J_\Theta \bigotimes_{i \in I(\Theta)} M_i(\xi_i) \right). \tag{6.23}$$

[2] Mathematically we require $X$ to be a complete separable metric space.

Since it is not obvious, we need to check well-definedness.

**Corollary 6.13.** *The triple* $\left(X_\Theta^0, \mathfrak{X}_\Theta^0, \nu_\rho\right)$ *constitutes a well defined probability space.*

*Proof.* To see that $(X_\Theta^0, \mathfrak{X}_\Theta^0)$ is a measurable space is trivial, since $X_\Theta^0$ is just a product of measurable spaces and inherits its measure structure from $\{0\} \cup X$, just like its $\sigma$-algebra $\mathfrak{X}_\Theta^0$. $\nu_\rho$ is indeed a probability measure, because it takes values in $[0, 1]$ and since $M_i$ is a POVM we easily calculate

$$\nu_\rho(\emptyset) = \operatorname{tr}\left(J_\Theta^\dagger \rho J_\Theta \bigotimes_{i \in I(\Theta)} M_i(\emptyset)\right) = \operatorname{tr}\left(J_\Theta^\dagger \rho J_\Theta \bigotimes_{i \in I(\Theta)} 0\right) = 0$$

and

$$\begin{aligned}
\nu_\rho\left(X_\Theta^0\right) = \nu_\rho\left(\prod_{i \in I(\Theta)} \{0\} \cup X\right) &= \operatorname{tr}\left(J_\Theta^\dagger \rho J_\Theta \bigotimes_{i \in I(\Theta)} M_i\left(\{0\} \cup X\right)\right) \\
&= \operatorname{tr}\left(J_\Theta^\dagger \rho J_\Theta \bigotimes_{i \in I(\Theta)} M_i\left(\{0\}\right) + M_i\left(X\right)\right) \\
&= \operatorname{tr}\left(J_\Theta^\dagger \rho J_\Theta \bigotimes_{i \in I(\Theta)} \mathbb{1}_\mathbb{C} \oplus \mathbb{1}_\mathcal{K}\right) = \operatorname{tr}\Big(J_\Theta^\dagger \rho J_\Theta\Big) = 1.
\end{aligned}$$

The $\sigma$-additivity of $\nu_\rho$ reduces mostly to the weak-$^*$ $\sigma$-additivity of the POVM. Indeed, having two disjoint sets of the form $\xi_i = \{\emptyset, \ldots, x_i, \ldots, \emptyset\}$ with $x_1$ and $x_2$ not at the same position, one easily sees that the trace splits into a sum. If $x_1$ and $x_2$ are located on the same position, we need $x_1 \cap x_2 = \emptyset$ in order to get disjoint sets $\xi_1$ and $\xi_2$. Hence we have

$$\begin{aligned}
\nu_\rho(\xi_1 \cup \xi_2) &= \operatorname{tr}\Big(J_\Theta^\dagger \rho J_\Theta \left[M_1(\emptyset) \otimes \cdots \otimes M_i(\{x_1 \cup x_2\}) \otimes \cdots \otimes M_i(\emptyset)\right]\Big) \\
&= \operatorname{tr}\Big(J_\Theta^\dagger \rho J_\Theta \left[M_1(\emptyset) \otimes \cdots \otimes M_i(\{x_1\}) + M_i(\{x_2\}) \otimes \cdots \otimes M_i(\emptyset)\right]\Big) \\
&= \nu_\rho(\xi_1) + \nu_\rho(\xi_2).
\end{aligned}$$

Since the sets $\xi_i$ clearly generate $\mathfrak{X}_\Theta^0$, it is sufficient to show the $\sigma$-additivity on this subset. So $\nu_\rho$ is a probability measure. □

We now have a probability space, but in order to define a point process we need a counting measure and insert a time-dependence into our formalism. We will define this point process step by step, starting with the kernel of its counting measure.

**Lemma 6.14.** *Let $\mathfrak{X}_{[0,T]}$ be the product $\sigma$-algebra of $[0,T] \times X$. The following map defines a transition kernel $\kappa$, counting the events falling in a given subset and which implies a counting measure on $[0,T] \times X$:*

$$\kappa : X^0_\Theta \times \mathfrak{X}_{[0,T]} \longrightarrow \mathbb{N}_0, \qquad \big((x_i)_{i\in I(\Theta)}, \xi\big) \longmapsto \sharp\left\{(t_{i-1}, x_i) \mid (t_{i-1}, x_i) \in \xi\right\}. \quad (6.24)$$

*If $X^0 \ni x_i = 0$, i.e. no event happened in the i-th subinterval, we see that $(t_{i-1}, x_i) \notin \xi$ for all $\xi \in \mathfrak{X}_{[0,T]}$, since $0 \notin X$.*

Here we chose to tag the event $x_i$ at the left boundary of its corresponding time interval $[t_{i-1}, t_i)$. We could have chosen any other point in this interval equivalently. So this definition can be physically motivated, we need to check, however, that it is well defined.

*Proof.* We proof that the map $\kappa$ defines a transition kernel between the measurable spaces $\big(X^0_\Theta, \mathfrak{X}^0_\Theta\big)$ and $\big([0,T] \times X, \mathfrak{X}_{[0,T]}\big)$.

**i)** For every fixed set $\xi \in \mathfrak{X}_{[0,T]}$ we have that the map

$$\kappa(\cdot, \xi) : \big(X^0_\Theta, \mathfrak{X}^0_\Theta\big) \longrightarrow (\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0)), \quad (x_i)_{i\in I(\Theta)} \longmapsto \sharp\left\{(t_{i-1}, x_i) \mid (t_{i-1}, x_i) \in \xi\right\}$$

is measurable, since every possible preimage of a $(\sharp I(\Theta))$-tuple of natural numbers (particle “clicks” per subinterval) can be easily constructed and is naturally an element of $\mathfrak{X}^0_\Theta$.

**ii)** Vice versa: Fixing an element $(x_i)_{i\in I(\Theta)} \in X^0_\Theta$ implies a map

$$\kappa\big((x_i)_{i\in I(\Theta)}, \cdot\big) : \mathfrak{X}_{[0,T]} \longrightarrow (\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0)), \quad \xi \longmapsto \sharp\left\{(t_{i-1}, x_i) \mid (t_{i-1}, x_i) \in \xi\right\}.$$

We now have to show that this map is a counting measure and do this straightforward by checking the defining properties: $\kappa\big((x_i)_{i\in I(\Theta)}, \cdot\big)$ is trivially non negative and it is also easy to see that $\kappa\big((x_i)_{i\in I(\Theta)}, \emptyset\big) = 0$, both independently of $(x_i)_{i\in I(\Theta)}$.

The only nontrivial part is the $\sigma$-additivity of mutually disjoint sets. Since $\mathfrak{X}_{[0,T]}$ is of form $\{\emptyset, \{([0,T], X)\}, \dots\}$, every $\xi \in \mathfrak{X}_{[0,T]}$ is a collection of tuples of subintervals (tagged via its left time boundary) and events happened in this interval. Sets are disjoint if and only if they don't share any common event per subinterval, because the "no-event" subintervals are not part of $[0, T] \times X$. But since the number of "clicks" are just summed by $\kappa$ for each input set, $\kappa$ splits into the sum of images of those disjoint sets. The map is therefore a counting measure. □

## Nets Of Point Processes

We now have a counting measure on $([0, T] \times X, \mathfrak{X}_{[0,T]})$ w.r.t. to certain choice of an element $(x_i)_{i \in I(\Theta)} \in X^0_\Theta$. Since a point process is defined as a counting measure valued random measure, we have everything we need in order to define such a map.

**Definition 6.15.** *Let $\Theta \in \mathfrak{Z}([0, T])$ and $\rho \in \mathfrak{T}(\mathcal{K})$ be a state. We then define the following point process:*

$$\begin{aligned} \widetilde{\mathfrak{P}}_\Theta : (X^0_\Theta, \mathfrak{X}^0_\Theta, \nu_\rho) &\longrightarrow \mathcal{CM}([0, T] \times X, \mathfrak{X}_{[0,T]}) \\ (x_i)_{i \in I(\Theta)} &\longmapsto \kappa_i(\,\cdot\,) := \kappa\left((x_i)_{i \in I(\Theta)}, \,\cdot\,\right). \end{aligned} \tag{6.25}$$

As outlined before, one can write every proper point process as the sum of Dirac measures. $\widetilde{\mathfrak{P}}_\Theta$ is proper, since $X$ was assumed to be at least a complete metric space. One easily sees that

$$\widetilde{\mathfrak{P}}_\Theta\left((x_i)_{i \in I(\Theta)}, \xi\right) = \sum_{n=1}^{\kappa_i(\xi)} \delta_{(t_{n-1}, x_n)} \tag{6.26}$$

is an equivalent formulation. As usual for stochastic processes, we will omit the first argument of $\widetilde{\mathfrak{P}}_\Theta$, since it will always be clear from context.

Considering nets we had the problem earlier that the image spaces of different operators, i.e. $\mathcal{K}_\Theta$, didn't coincide. We therefore embedded all the image spaces into a common Hilbert space $\mathcal{K}_{[0,T]}$. We now have the situation where every point process has a $\mathfrak{Z}([0, T])$ dependent domain. If we want to have the same probability space for all possible discretizations, and therefore define point processes on a common domain for all interval decompositions, we can use the following embedding $\Upsilon$ and push the measure $\nu_\rho$ forward along it.

$$\Upsilon : X^0_\Theta \longrightarrow X_{\text{com}} := \bigcup_{n=0}^{\infty} \Delta_n([0, T]) \times X^n,$$

$$(0,\ldots,\underset{\substack{\uparrow\\ \text{pos. } i_1}}{x_1},0,\ldots,0,\underset{\substack{\uparrow\\ \text{pos. } i_n}}{x_n},\ldots,0) \longmapsto \left(t_{i_1-1},\ldots,t_{i_n-1}\right) \times (x_1,\ldots,x_n)\,, \tag{6.27}$$

where $\Delta_n([0,T])$ denotes the set of ordered n-tuples in $[0,T]$. One is then able to define the probability space $\left(X_{\text{com}}, \mathfrak{X}_{\text{com}}, \Upsilon_*\left(\nu_\rho\right)\right)$, with the Borel $\sigma$-algebra $\mathfrak{X}_{\text{com}}$. A point process on $X_{\text{com}}$ is then given by

$$\begin{aligned} \mathfrak{P}_\Theta : \left(X_{\text{com}}, \mathfrak{X}_{\text{com}}, \Upsilon_*\left(\nu_\rho\right)\right) &\longrightarrow \mathcal{CM}([0,T]\times X, \mathfrak{X}_{[0,T]}) \\ \Upsilon\left((x_i)_{i\in I(\Theta)}\right) &\longmapsto \widetilde{\mathfrak{P}}_\Theta\left((x_i)_{i\in I(\Theta)},\,\cdot\,\right). \end{aligned} \tag{6.28}$$

Heuristically taking a refinement limit now should be seen as a net of point processes $\Theta \longmapsto \mathfrak{P}_\Theta$ converging to a Poisson process. We continue further elaborating this idea. But before we do this, we need to talk about the subtle relation between characteristic functions and expectation values of Weyl operators.

## 6.5 Characteristic Functions as Expectation Values of Weyl Operators

Let us recall that the discrete field operators w.r.t. a function $|\lambda\rangle \in L^2([0,T],\mathcal{K})$ were defined on every subinterval of $\Theta \in \mathfrak{Z}([0,T])$ as

$$\Phi : \mathcal{K} \longrightarrow \mathcal{B}(\mathbb{C}\oplus\mathcal{K}), \quad \Phi\left(\sqrt{\tau_i}\,|\overline{\lambda}_i\rangle\right) = \begin{pmatrix} 0 & \sqrt{\tau_i}\,\langle\overline{\lambda}_i| \\ -\sqrt{\tau_i}\,|\overline{\lambda}_i\rangle & 0 \end{pmatrix} \tag{6.29}$$

and the Weyl operator was the operator exponential, tensored over every $i \in I(\Theta)$. Note that, since $\Phi$ is skew Hermitian, we omitted the usual imaginary $i$ in the definition of the Weyl operator.

If we want to learn about the stochastic process corresponding to the discrete Weyl operators, we need to look at the POVM (or in this case PVM) which canonically belongs to the operator $\Phi_\Theta(|\lambda\rangle)$. Since $i\Phi_\Theta(|\lambda\rangle)$ is a self-adjoint operator[3], it has well defined spectral projections. Those projections canonically imply a PVM with values in $\mathcal{B}(\mathbb{C}\oplus\mathcal{K})$.

[3] Actually this even holds for each of the operators $i\Phi(\sqrt{\tau_i}\,|\overline{\lambda}_i\rangle)$.

**Corollary 6.16.** *Given* $|\lambda\rangle \in L^2([0,T],\mathcal{K})$ *we see that the spectral projections of* $\Phi\left(\sqrt{\tau_i}\,\left|\overline{\lambda}_i\right\rangle\right)$ *are given by:*

$$M_+\left(\left|\overline{\lambda}_i\right\rangle\right) \in \mathcal{B}(\mathbb{C}\oplus\mathcal{K}), \qquad M_+\left(\left|\overline{\lambda}_i\right\rangle\right) := \frac{1}{2}\begin{pmatrix} 1 & \langle e_i| \\ -|e_i\rangle & |e_i\rangle\!\langle e_i| \end{pmatrix}, \tag{6.30}$$

$$M_-\left(\left|\overline{\lambda}_i\right\rangle\right) \in \mathcal{B}(\mathbb{C}\oplus\mathcal{K}), \qquad M_-\left(\left|\overline{\lambda}_i\right\rangle\right) := \frac{1}{2}\begin{pmatrix} 1 & -\langle e_i| \\ |e_i\rangle & |e_i\rangle\!\langle e_i| \end{pmatrix}, \tag{6.31}$$

$$M_0\left(\left|\overline{\lambda}_i\right\rangle\right) \in \mathcal{B}(\mathbb{C}\oplus\mathcal{K}), \qquad M_0\left(\left|\overline{\lambda}_i\right\rangle\right) := \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}-|e_i\rangle\!\langle e_i| \end{pmatrix}, \tag{6.32}$$

*where* $|e_i\rangle$ *is the unit vector in direction* $\left|\overline{\lambda}_i\right\rangle$. *Clearly all those operators are projections and we can recover* $\Phi$ *by*

$$\Phi\left(\sqrt{\tau_i}\,\left|\overline{\lambda}_i\right\rangle\right) = \sqrt{\tau_i}\,\|\overline{\lambda}_i\|M_+\left(\left|\overline{\lambda}_i\right\rangle\right) - \sqrt{\tau_i}\,\|\overline{\lambda}_i\|M_-\left(\left|\overline{\lambda}_i\right\rangle\right). \tag{6.33}$$

*Proof.* A quick calculation shows:

$$\begin{aligned}
&\sqrt{\tau_i}\,\|\overline{\lambda}_i\|M_+\left(\left|\overline{\lambda}_i\right\rangle\right) - \sqrt{\tau_i}\,\|\overline{\lambda}_i\|M_-\left(\left|\overline{\lambda}_i\right\rangle\right) \\
&= \frac{\sqrt{\tau_i}}{2}\|\overline{\lambda}_i\|\begin{pmatrix} 1 & \langle e_i| \\ -|e_i\rangle & |e_i\rangle\!\langle e_i| \end{pmatrix} - \frac{\sqrt{\tau_i}}{2}\|\overline{\lambda}_i\|\begin{pmatrix} 1 & -\langle e_i| \\ |e_i\rangle & |e_i\rangle\!\langle e_i| \end{pmatrix} \\
&= \sqrt{\tau_i}\,\|\overline{\lambda}_i\|\begin{pmatrix} 0 & \langle e_i| \\ -|e_i\rangle & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\tau_i}\,\langle\overline{\lambda}_i| \\ -\sqrt{\tau_i}\,|\overline{\lambda}_i\rangle & 0 \end{pmatrix} \\
&= \Phi\left(\sqrt{\tau_i}\,\left|\overline{\lambda}_i\right\rangle\right),
\end{aligned}$$

which proves the corollary. □

Notice that $I(\Theta) \ni i \longmapsto M_i := \left\{M_+\left(\left|\overline{\lambda}_i\right\rangle\right), M_-\left(\left|\overline{\lambda}_i\right\rangle\right), M_0\left(\left|\overline{\lambda}_i\right\rangle\right)\right\}$ is a PVM on $\mathcal{B}(\mathbb{C}\oplus\mathcal{K})$ for every $i \in I(\Theta)$, because those operators sum to the identity. Every field operator on a single time-step therefore canonically induces a measurement via this PVM with outcome space $X^0 = \{+, -, 0\}$. This can be interpreted as a kind of homodyne detection, as seen in [Neu15][p. 145 et seq.].

If we want to identity this PVM on the i-th subinterval with the corresponding quantum field operator, one has to let the PVM act on a certain function $f_i : \mathfrak{X}^0 \longrightarrow \mathbb{R}$. If we fix $|\lambda\rangle \in L^2([0,T], \mathcal{K})$ we define $f$ on the three "generating" sets of $\mathfrak{X}^0$ as

$$f_i\left(\{+\}\right) := f_i^+ = \sqrt{\tau_i}\left\|\overline{\lambda}_i\right\|, \quad f_i(\{-\}) := f_i^- = -\sqrt{\tau_i}\left\|\overline{\lambda}_i\right\|, \quad f_i(\{0\}) := f_i^0 = 0.$$

One can then easily see that $M_i[f] := \Phi\left(\sqrt{\tau_i}\left|\overline{\lambda}_i\right\rangle\right)$. If we want a PVM describing the whole time interval $[0,T]$ we lift, as before, the domain to the Cartesian product

$$X_\Theta^0 := \prod_{i \in I(\Theta)} \{+, -, 0\} \tag{6.34}$$

and define the PVM $M : \mathfrak{X}_\Theta^0 \longrightarrow \mathcal{B}(\mathcal{K}_\Theta)$ in the obvious way, as $M_i$ acting on the i-th factor and imposing linearity. Here $\mathfrak{X}_\Theta^0$ clearly denotes the $\sigma$-algebra of $X_\Theta^0$

We have now seen how every quantum field operator, for a given $|\lambda\rangle$, canonically defines a measurement in the form of a PVM. In the beginning of this chapter, we have seen how every POVM on $\mathcal{B}(\mathcal{K})$ (or $\mathcal{B}(\mathbb{C} \oplus \mathcal{K})$) induces a point process corresponding to this measurement setup. Before combining those two notions, we need to rewrite our Weyl operator using the $\Gamma_\Theta$ functor.

**Corollary 6.17.** *Using the second quantization notation we can equivalently reformulate the definition of the Weyl operator from before as:*

$$W_\Theta\left(|\lambda\rangle\right) = \exp(\Phi_\Theta\left(|\lambda\rangle\right)) = \exp\!\Big(\Gamma_\Theta\left(\Phi\left(\sqrt{\tau_i}\left|\overline{\lambda}_i\right\rangle\right)\right)\Big). \tag{6.35}$$

*Proof.* We insert the definitions and use the linearity of the tensor product to calculate:

$$\begin{aligned} W_\Theta\left(\left|\overline{\lambda}_i\right\rangle\right) &:= \bigotimes_{i \in I(\Theta)} \exp\!\big(\Phi\left(\sqrt{\tau_i}\left|\overline{\lambda}_i\right\rangle\right)\big) \\ &= \prod_{i \in I(\Theta)} \mathbb{1} \otimes \cdots \otimes \exp\!\big(\Phi\left(\sqrt{\tau_i}\left|\overline{\lambda}_i\right\rangle\right)\big) \otimes \cdots \otimes \mathbb{1} \end{aligned}$$

$$
\begin{aligned}
&= \prod_{i\in I(\Theta)} \mathbb{1}\otimes\cdots\otimes\sum_{n=0}^{\infty}\frac{\Phi\left(\sqrt{\tau_i}\left|\overline{\lambda}_i\right\rangle\right)^n}{n!}\otimes\cdots\otimes\mathbb{1}\\
&= \prod_{i\in I(\Theta)}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\mathbb{1}\otimes\cdots\otimes\Phi\left(\sqrt{\tau_i}\left|\overline{\lambda}_i\right\rangle\right)\otimes\cdots\otimes\mathbb{1}\right)^n\\
&= \prod_{i\in I(\Theta)}\exp\left(\mathbb{1}\otimes\cdots\otimes\Phi\left(\sqrt{\tau_i}\left|\overline{\lambda}_i\right\rangle\right)\otimes\cdots\otimes\mathbb{1}\right)\\
&= \exp\left(\sum_{i\in I(\Theta)}\mathbb{1}\otimes\cdots\otimes\Phi\left(\sqrt{\tau_i}\left|\overline{\lambda}_i\right\rangle\right)\otimes\cdots\otimes\mathbb{1}\right)\\
&= \exp\Big(\Gamma_\Theta\left(\Phi\left(\sqrt{\tau_i}\left|\overline{\lambda}_i\right\rangle\right)\right)\Big) = \exp(\Phi_\Theta\left(|\lambda\rangle\right)).
\end{aligned}
$$

And so the corollary is proven. □

As outlined before, we are now able to combine the quantum field operators with the characteristic function, induced by its canonical PVM, via expectation values of the Weyl operator.

**Theorem 6.18.** *The expectation value of the Weyl operator $W(|\lambda\rangle)$ is the characteristic function of the point process $\mathfrak{P}_\Theta$ associated with the PVM $M$ and function $f$ corresponding to the quantum field $\Phi$ generating $W(|\lambda\rangle)$, i.e.*

$$
\langle W(|\lambda\rangle)\rangle_\rho = C(f) = \varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \operatorname{tr}\Big(\rho\, J_\Theta\left(\Gamma_\Theta M\right)\left[e^{i\mathfrak{P}_\Theta[f]}\right] J_\Theta^\dagger\Big). \qquad (6.36)
$$

*The limit is meant to be in the weak-* topology.*

*Proof.* Let $\Theta\in\mathfrak{Z}([0,T])$ and $\rho\in\mathfrak{T}(\mathcal{K}_{[0,T]})$ be a state. We can be rewrite $\rho$ in the form

$$
\rho = \varinjlim_{\Theta\in\mathfrak{Z}([0,T])} J_\Theta\rho_\Theta J_\Theta^\dagger,
$$

with $\Theta\longmapsto\rho_\Theta\in\mathcal{B}(\mathcal{K}_\Theta)$ being a Cauchy net. The expectation value of $W(|\lambda\rangle)$ is then given by

$$\begin{aligned}\langle W(|\lambda\rangle)\rangle_\rho &= \operatorname{tr}(\rho\, W(|\lambda\rangle)) = \operatorname{tr}\left(\varinjlim_{\Theta\in\mathfrak{Z}([0,T])} J_\Theta \rho_\Theta W_\Theta\, |\lambda\rangle\, J_\Theta^\dagger\right)\\ &= \varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \operatorname{tr}\Big(J_\Theta \rho_\Theta W_\Theta\, |\lambda\rangle\, J_\Theta^\dagger\Big) = \varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \operatorname{tr}(\rho_\Theta W_\Theta\, |\lambda\rangle).\end{aligned}$$

So we can restrict ourselves to analyzing the discrete expectation values. In order to do so, we need to show the following identity first. Let $M : \mathfrak{X}_\Theta^0 \longrightarrow \mathcal{B}(\mathcal{K}_\Theta)$ be the canonical PVM with function $f$ corresponding to the quantum field $\Phi$, generating the Weyl operator. Due to the linearity of the PVM we look at the i-th subinterval of $\Theta \in \mathfrak{Z}([0,T])$ and calculate the action of $M_i$ on $e^{if}$, i.e.

$$\begin{aligned}M_i\left[e^{if}\right] &= e^{if_i^+} M_+\left(\left|\overline{\lambda}_i\right\rangle\right) + e^{if_i^-} M_-\left(\left|\overline{\lambda}_i\right\rangle\right) + e^{if_i^0} M_0\left(\left|\overline{\lambda}_i\right\rangle\right)\\ &= \begin{pmatrix} \frac{1}{2}e^{if_i^+} + \frac{1}{2}e^{if_i^-} & \left(\frac{1}{2}e^{if_i^+} - \frac{1}{2}e^{if_i^-}\right)\langle e_i| \\ \left(-\frac{1}{2}e^{if_i^+} + \frac{1}{2}e^{if_i^-}\right)|e_i\rangle & \mathbb{1}_\mathcal{K} + |e_i\rangle\!\langle e_i| \left(\frac{1}{2}e^{if_i^+} + \frac{1}{2}e^{if_i^-} - 1\right)\end{pmatrix}\\ &= \begin{pmatrix} \cos(\theta_i) & \sin(\theta_i)\,\langle e_i| \\ -\sin(\theta_i)\,|e_i\rangle & \mathbb{1}_\mathcal{K} + (\cos(\theta_i) - 1)\,|e_i\rangle\!\langle e_i|\end{pmatrix} = (5.14)\\ &= \exp\Big(\Phi\left(\sqrt{\tau_i}\,\left|\overline{\lambda}_i\right\rangle\right)\Big) = e^{M_i[f]}.\end{aligned}$$

One therefore has $M\left[e^{if}\right] = e^{M[f]}$ on all of $X_\Theta^0$, since the calculation will always split linear into the subintervals. We now have everything we need to prove this theorem. For convenience, let us collect the following identities, which we already used in this thesis.

I) Equation (6.35), i.e. $W_\Theta(|\lambda\rangle) = \exp(\Phi_\Theta(|\lambda\rangle)) = \exp\Big(\Gamma_\Theta\left(\Phi\left(\sqrt{\tau_i}\,\left|\overline{\lambda}_i\right\rangle\right)\right)\Big)$.

II) $\Phi\left(\sqrt{\tau_i}\,\left|\overline{\lambda}_i\right\rangle\right) = M_i[f]$ as seen in equation (6.33).

III) $\Gamma_\Theta(\exp(A)) = \exp(\Gamma_\Theta(A))$ for every $A \in \mathcal{B}(\mathbb{C}\oplus\mathcal{K})$, which should be obvious due to the multi-linearity of the tensor product.

IV) $\exp(M[f]) = M\left[e^{if}\right]$, as seen above.

V) $(\Gamma_\Theta M)\left[e^{i\mathfrak{P}_\Theta[f]}\right] = \Gamma_\Theta\left(M\left[e^{if}\right]\right)$ from equation (6.17).

VI) $C_\Theta(f) = \mathrm{tr}\Big(\rho J_\Theta \left(\Gamma_\Theta M\right) \left[e^{i\mathfrak{P}_\Theta[f]}\right] J_\Theta^\dagger\Big) = \mathrm{tr}\left(\underbrace{J_\Theta^\dagger \rho J_\Theta}_{=\rho_\Theta} \left(\Gamma_\Theta M\right) \left[e^{i\mathfrak{P}_\Theta[f]}\right]\right)$ as outlined in equation (6.18).

If we now analyze the expectation value of our discrete Weyl operator, i.e. fixing an interval decomposition $\Theta \in \mathfrak{Z}([0,T])$, we have

$$\begin{aligned}\langle W_\Theta(|\lambda\rangle)\rangle_{\rho_\Theta} &\overset{\text{Def.}}{=} \mathrm{tr}(\rho_\Theta W_\Theta(|\lambda\rangle)) \overset{\text{I}}{=} \mathrm{tr}\Big(\rho_\Theta \exp\Big(\Gamma_\Theta\left(\Phi\left(\sqrt{\tau_i}\,\big|\overline{\lambda}_i\rangle\right)\right)\Big)\Big) \\ &\overset{\text{II}}{=} \mathrm{tr}(\rho_\Theta \exp(\Gamma_\Theta\left(M[f]\right))) \overset{\text{III}}{=} \mathrm{tr}(\rho_\Theta \Gamma_\Theta\left(\exp(M[f])\right)) \\ &\overset{\text{IV}}{=} \mathrm{tr}\Big(\rho_\Theta \Gamma_\Theta\left(M\left[e^{if}\right]\right)\Big) \overset{\text{V}}{=} \mathrm{tr}\Big(\rho_\Theta\left(\Gamma_\Theta M\right)\left[e^{i\mathfrak{P}_\Theta[f]}\right]\Big) \\ &\overset{\text{VI}}{=} C_\Theta(f).\end{aligned}$$

Since $W_\Theta$ converges strongly, one is able to perform the weak-* limit $\varinjlim_{\Theta \in \mathfrak{Z}([0,T])}$. ☐

## 6.6 Nets of Characteristic Functions

We now have a point process, depending on $\Theta \in \mathfrak{Z}([0,T])$, which should model beam-type measurements in laboratories (and therefore continuous measurements) quite accurately. Convergence of those processes can be physically motivated, but is not mathematically defined a priori on the level of point processes.

For every $\Theta \in \mathfrak{Z}([0,T])$ we are able to construct characteristic functions $C_\Theta(f)$ for each of those point processes in a canonical way as seen above. In order to define convergence of those point processes it seems to be a natural idea to define convergence with respect to their characteristic functions, since those functions inherit all the information encoded in the point process.

We have defined point processes corresponding to a PVM coming from the quantum field itself. In this analysis however, we will be more general, constructing characteristic functions in the case of a general one-particle time POVM $F$ with outcome space $X$.
Let us assume, for the moment, that $F$ is independent of each time-step, i.e. $F_i = F$ for all $i \in I(\Theta)$. Physically this would correspond to the situation where we always perform "the same" measurement on our physical system.

**Theorem 6.19.** *Let $\rho \in \mathfrak{T}\left(\mathcal{K}_{[0,T]}\right)$ be a state and let $F : \mathfrak{X} \longrightarrow \mathcal{B}(\mathcal{K})$ be a POVM. For a continuous function $f : [0,T] \times X \longrightarrow \mathbb{R}$, with $f_j(x) := f(t_j, x)$, we write*

$$F\left[e^{if_j}\right] = \int\limits_X \exp\big(if(t_j,x)\big)F(\mathrm{d}x) \in \mathcal{B}(\mathcal{K}). \tag{6.37}$$

*For every $\Theta \in \mathfrak{Z}([0,T])$ the function $C_\Theta(f)$, defined as*

$$\begin{aligned} C_\Theta &: C([0,T] \times X, \mathbb{R}) \longrightarrow \mathbb{C} \\ f &\longmapsto C_\Theta(f) = \operatorname{tr}\left(J_\Theta^\dagger \rho J_\Theta \Gamma_\Theta \begin{pmatrix} 1 & 0 \\ 0 & F\left[e^{if_j}\right] \end{pmatrix}\right), \end{aligned} \tag{6.38}$$

*is the characteristic function of the point process $\mathfrak{P}_\Theta$ and measured by the canonical arrival time POVM $M$. Furthermore, the net of point processes converges in the sense that their characteristic functions converge weak-*.*

Note that in this theorem we implicitly lifted the domain of $f$ to $[0,T] \times X^0$ by setting $f(t,0) := 0 \; \forall t \in [0,T]$, in order to be measured by $M$ accurately.

*Proof.* The well definedness and convergence of $\Theta \longmapsto C_\Theta(f)$ is a direct consequence of theorem 5.5. The net $\Theta \longmapsto \Gamma_\Theta\left(\mathbb{1}_\mathbb{C} \otimes F\left[e^{if_j}\right]\right)$ converges therefore strongly for all $f$ and hence $C_\Theta(f)$ converges at least weak-* for every $f$.

Since $F_i = F$ for all $i \in I(\Theta)$ and $M_i(\{0\}) = M(\{0\})$, we can omit every index on our POVMs and see the correspondence of $M$ and $F$ via

$$\begin{aligned} M\left[e^{if_j}\right] &= \int\limits_{X^0} e^{if_j(x)}M(\mathrm{d}x) = e^{if_j(0)}M(\{0\}) + \int\limits_X e^{if_j(x)}M(\mathrm{d}x) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \int_X e^{if_j(x)}F(\mathrm{d}x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & F\left[e^{if_j}\right] \end{pmatrix}. \end{aligned}$$

Looking at the definition of a characteristic function of the point process $\mathfrak{P}_\Theta$ w.r.t. the measurement $M$, we see

$$
\begin{aligned}
C_\Theta(f) &\overset{(6.11)}{:=} \operatorname{tr}\left(\rho\, J_\Theta\left(\Gamma_\Theta M\right)\left[e^{i\mathfrak{P}_\Theta[f]}\right] J_\Theta^\dagger\right) \\
&= \operatorname{tr}\left(J_\Theta^\dagger \rho J_\Theta\left(\Gamma_\Theta M\right)\left[e^{i\mathfrak{P}_\Theta[f]}\right]\right) \\
&\overset{(6.18)}{=} \operatorname{tr}\left(J_\Theta^\dagger \rho J_\Theta\,\Gamma_\Theta\left(M\left[e^{if_j}\right]\right)\right) \\
&= \operatorname{tr}\left(J_\Theta^\dagger \rho J_\Theta\,\Gamma_\Theta\begin{pmatrix}1 & 0\\ 0 & F\left[e^{if_j}\right].\end{pmatrix}\right) = (6.38),
\end{aligned}
$$

which proves the theorem. □

Taking the right continuity conditions we can also take the involved POVMs to be time dependent and the functions $f$ to be complex valued. We then arrive at a description of counting statistics for point processes measured by general second quantized POVM $F$.

**Theorem 6.20.** *Let $\rho \in \mathfrak{T}(\mathcal{K}_{[0,T]})$ be a state and $F : \mathfrak{X}_{[0,T]} \longrightarrow \mathcal{B}(\mathcal{K})$ be a time dependent POVM. Given a measurable function $f : [0,T] \times X \longrightarrow \mathbb{C}$ and $\Theta \in \mathfrak{Z}([0,T])$ we have for every $i \in I(\Theta)$*

$$
\int\limits_{[t_{i-1},t_i]\times X} \exp(if(t,x))F(\mathrm{d}t,\mathrm{d}x) = F\left[e^{if}\right] \in \mathcal{B}(\mathcal{K}). \tag{6.39}
$$

*The following function is the characteristic functions of the point process $\mathfrak{P}_\Theta$ w.r.t. to general POVMs $F$ for every $\Theta \in \mathfrak{Z}([0,T])$:*

$$
\begin{aligned}
C_\Theta &: L^0([0,T]\times X,\mathbb{C}) \longrightarrow \mathbb{C}, \\
f &\longmapsto C_\Theta(f) := \operatorname{tr}\left(J_\Theta^\dagger \rho J_\Theta\,\Gamma_\Theta\begin{pmatrix}1 & 0\\ 0 & \frac{1}{\tau_i}F\left[e^{if}\right]\end{pmatrix}\right).
\end{aligned} \tag{6.40}
$$

Here $L^0([0,T]\times X,\mathbb{R})$ denotes the set of measurable functions. The convergence of the net $\Theta \longmapsto C_\Theta(f)$ then depends on the specific choice of $f$ and $F$. Also note that the characteristic function might evaluate to $\infty$, since we didn't require integrability of $f$.

# Chapter 7

# cMPS and Outlook

In this chapter we present the major theorems of [Neu15], mainly the construction and convergence of continuous Stinespring dilations. We are able to use the results discovered prior in this thesis in combination with those continuous dilations to define cMPS, obtain explicit Lindblad equations and find intricate connections to gauge processes. We end with an outlook containing multiple possibilities to solve the upcoming PDEs and calculate an explicit example.

## 7.1 cMPS via Repeated Quantum Observations

We start by collecting the definitions and theorems needed in order define continuous Stinespring dilations.

### Continuous Stinespring Dilations

The following theorems are split into a bounded and an unbounded version, since the corresponding convergence conditions and necessary assumptions are sufficiently different.

**Unbounded assumptions:** Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $\mathcal{D}$ a Banach space which can be densely and continuously embedded into $\mathcal{H}$. Also let $s \leq t \in [0, T]$. Furthermore:

U.I. Let $U(t, s) : \mathcal{H} \supset \mathcal{D} \longrightarrow \mathcal{H}$ be a contractive evolution system with common core $\mathcal{D}$ and generator $K$.

U.II. Let $L(t) : \mathcal{H} \longrightarrow \mathcal{K} \otimes \mathcal{H}$ be a family of operators fulfilling the following two properties:

a) $L(t) : \mathcal{D} \longrightarrow \mathcal{K} \otimes \mathcal{H}$ is a family of operators s.t. $\forall \left|\psi\right\rangle \in \mathcal{D}$ the function $t \longmapsto L(t) \left|\psi\right\rangle$ is piece-wise continuous.

b) Analogously to equation (2.15) we have that for all $\left|\psi\right\rangle \in \mathcal{D}$ the infinitesimal conservativity condition

$$\lim_{h\to 0} \frac{1}{h} \|U(t+h,t) \left|\psi\right\rangle\|^2 + \|L(t) \left|\psi\right\rangle\|^2 \leq 0 \tag{7.1}$$

holds.

The typical example for such a system would be that $L(t)$ is time independent and that $U(t,s) = \exp((t-s)K)$ holds, i.e. $U(t,s)$ would constitute a strongly continuous semigroup with generator $K : \mathcal{H} \supset \mathrm{dom}(K) \longrightarrow \mathcal{H}$. Together with the condition (7.1), this would be equivalent to $\mathcal{L}(B)$ being a standard Lindblad generator given by

$$\langle \varphi | \mathcal{L}(B) \psi \rangle = \langle K\varphi | B\psi \rangle + \langle \varphi | BK\psi \rangle + \langle L(t)\varphi | (\mathbb{1}_{\mathcal{K}} \otimes B)\, L(t)\psi \rangle\,, \tag{7.2}$$

with $B \in \mathcal{B}(\mathcal{H})$ and $\left|\varphi\right\rangle, \left|\psi\right\rangle \in \mathcal{H}$.

However, if we consider bounded generators $L$ and $K$, we can modify the assumptions in the following way.

**Bounded assumptions** Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and let $s \leq t \in [0,T]$. Furthermore let $K(t) \in \mathcal{B}(\mathcal{H})$ and $L(t) : \mathcal{H} \longrightarrow \mathcal{K} \otimes \mathcal{H}$ be two families of bounded operators, s.t.

B.I. The function $t \longmapsto K(t)$ is continuous and $\|K(t)\| \leq C$ for $t \in [0,T]$.

B.II. The function $t \longmapsto L(t)$ is continuous.

B.III. For every $t \in [0,T]$ we have that

$$K^\dagger(t) + K(t) + L(t)^\dagger L(t) \leq 0. \tag{7.3}$$

These sets of assumptions are necessary to define the following important maps.

**Definition 7.1.** *Let* $|\varphi\rangle \in \mathcal{D}$ *and* $\Theta \in \mathfrak{Z}([0,T])$. *We then define a map* $V_i : \mathcal{H} \longrightarrow (\mathbb{C} \oplus \mathcal{K}) \otimes \mathcal{H}$ *for all* $i \in I(\Theta)$ *piece wise via:*

$$\begin{aligned}
V_{i,0} &: \mathcal{H} \longrightarrow \mathcal{H}, && |\varphi\rangle \longmapsto U(t_i, t_{i-1})\,|\varphi\rangle \\
V_{i,1} &: \mathcal{H} \longrightarrow \mathcal{K} \otimes \mathcal{H}, && |\varphi\rangle \longmapsto \frac{1}{\sqrt{\tau_i}} \int\limits_{t_{i-1}}^{t_i} \mathbb{1} \otimes U(t_i, s) L(s) U(s, t_{i-1})\,|\varphi\rangle \;\mathrm{d}s \\
V_i &: \mathcal{H} \longrightarrow (\mathbb{C} \oplus \mathcal{K}) \otimes \mathcal{H}, && |\varphi\rangle \longmapsto (V_{i,0}\,|\varphi\rangle\,, V_{i,1}\,|\varphi\rangle).
\end{aligned}$$

*In this situation we define the* $\sharp I(\Theta)$*-times concatenated map* $V_\Theta$ *as*

$$V_\Theta : \mathcal{H} \longrightarrow \mathcal{K}_\Theta \otimes \mathcal{H}, \qquad V_\Theta := \prod_{i \in I(\Theta)} V_i. \tag{7.4}$$

The convergence of these maps in highly nontrivial and subject of the folling theorem.

**Theorem 7.2.** *The following two constructions are well defined.*

***Unbounded Case:*** *Under the "unbounded assumptions" the net* $\Theta \longmapsto (J_\Theta \otimes \mathbb{1}_\mathcal{H})V_\Theta$ *converges strongly for* $\Theta \in \mathfrak{Z}([s,t])$. *In other words:* $\forall s \leq t \in [0,T]$, $\forall\, |\varphi\rangle \in \mathcal{H}, \forall \varepsilon > 0\; \exists\, \Theta \in \mathfrak{Z}([s,t])$ *s.t.*

$$\|(V_\Lambda - (J_{\Lambda\Xi} \otimes \mathbb{1}_\mathcal{H})\,V_\Xi)\,|\varphi\rangle\| \leq \varepsilon \quad \forall\; \Theta \subseteq \Xi \subseteq \Lambda \in \mathfrak{Z}([s,t]). \tag{7.5}$$

***Bounded Case:*** *Under the "bounded assumptions" the net* $\Theta \longmapsto (J_\Theta \otimes \mathbb{1}_\mathcal{H})V_\Theta$ *for* $\Theta \in \mathfrak{Z}([s,t])$ *converges in the norm topology. In other words:* $\forall s \leq t \in [0,T]$, $\forall \varepsilon > 0\; \exists\, \Theta \in \mathfrak{Z}([s,t])$ *s.t.*

$$\|V_\Lambda - (J_{\Lambda\Xi} \otimes \mathbb{1}_\mathcal{H})\,V_\Xi\| \leq \varepsilon \qquad \forall\; \Theta \subseteq \Xi \subseteq \Lambda \in \mathfrak{Z}([s,t]). \tag{7.6}$$

Having shown the respective convergence one can perform the net limit and define continuous Stinespring dilations.

**Definition 7.3.** *The limit of the* $\Theta \longmapsto V_\Theta$ *constitutes a* ***continuous Stinespring dilation*** *and the following maps* $\mathbb{E}(s,t)$ *and* $\widehat{\mathbb{E}}(s,t)$ *can be motivated to be* ***continuous quantum channels*** *compatible with measurements.*

$$V_{[s,t]} : \mathcal{H} \longrightarrow \mathcal{K}_{[s,t]} \otimes \mathcal{H} \qquad |\varphi\rangle \longmapsto \varinjlim_{\Theta \in \mathfrak{Z}([s,t])} (J_\Theta \otimes \mathbb{1}_\mathcal{H}) V_\Theta |\varphi\rangle \tag{7.7}$$

$$\mathbb{E}(s,t) : \mathcal{B}(\mathcal{K}_{[s,t]} \otimes \mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}) \qquad X \longmapsto V_{[s,t]}^\dagger X V_{[s,t]} \tag{7.8}$$

$$\widehat{\mathbb{E}}(s,t) : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}) \qquad B \longmapsto \mathbb{E}(s,t)\left(\mathbb{1}_{\mathcal{K}_{[s,t]}} \otimes B\right) \tag{7.9}$$

**Theorem 7.4.** *The quantum channels $\widehat{\mathbb{E}}(s,t)$ are, seen mathematically, evolution systems and obey a certain Cauchy equation, as described in equation (2.16).*

***Unbounded Case:*** *The family of maps $\widehat{\mathbb{E}}(s,t)$ for $0 \leq s \leq t \leq T$ is a weak-$*$ continuous evolution system of completely positive maps and a minimal solution to the Cauchy equation*

$$\frac{\mathrm{d}}{\mathrm{d}s} \left\langle \psi \middle| \widehat{\mathbb{E}}(s,T)(B)\varphi \right\rangle = \left\langle \psi \middle| \mathcal{L}(s)\widehat{\mathbb{E}}(s,T)(B)\varphi \right\rangle \tag{7.10}$$

*with generator*

$$\langle \psi | \mathcal{L}(t)(B)\varphi \rangle = \langle K(t)\psi | B\varphi \rangle + \langle \psi | BK(t)\varphi \rangle + \langle L(t)\psi | (\mathbb{1}_\mathcal{K} \otimes B)\, L(t)\varphi \rangle . \tag{7.11}$$

***Bounded Case:*** *The family of maps $\widehat{\mathbb{E}}(s,t)$ for $0 \leq s \leq t \leq T$ is a norm continuous evolution system of normal and completely positive maps and a minimal solution to the Cauchy equation*

$$\frac{\mathrm{d}}{\mathrm{d}s} \widehat{\mathbb{E}}(s,T)(B) = \mathcal{L}(s)\widehat{\mathbb{E}}(s,T)(B) \tag{7.12}$$

*with generator*

$$\mathcal{L}(s)(B) = K(s)^\dagger B + BK(s) + L(s)^\dagger (\mathbb{1}_\mathcal{K} \otimes B)\, L(s). \tag{7.13}$$

## Defining cMPS

Having a continuous quantum channel $\mathbb{E}_{[0,T]}$ we canonically get a continuous analogue of a matrix product state as outlined in the end of chapter 4. It seems natural to define the following cMPS.

**Definition 7.5.** *Under the "unbounded assumptions" let $B \in \mathcal{B}(\mathcal{H})$ be a bounded operator containing information about the boundary conditions and let $\rho \in \mathfrak{T}(\mathcal{H})$ be a state. A **cMPS** $\omega$ is then defined as $\omega = \|\widetilde{\omega}\|^{-1}\widetilde{\omega}$ with*

$$\begin{aligned} \widetilde{\omega} : \mathcal{B}(\mathcal{K}_{[0,T]}) &\longrightarrow \mathbb{C} \\ W &\longmapsto \operatorname{tr}\big(\rho\ \mathbb{E}_{[0,T]}(W \otimes B)\big). \end{aligned} \tag{7.14}$$

Here, as notation may has implied, the operator $W$ is meant to be a Weyl operator, since we know that every possible element in $\mathcal{B}(\mathcal{K}_{[0,T]})$ has to be of this form.

The Weyl operator depends, for our purpose, on three different arguments, i.e. $W = W(c, |\lambda\rangle, U)$. Given a quantum state $\rho \in \mathfrak{T}(\mathcal{H})$ with boundary condition $B \in \mathcal{B}(\mathcal{H})$ one can see this cMPS, in analogy to a Wightman field, as a distribution of the form

$$C_c^\infty([0,T], \mathcal{K}) \ni |\lambda\rangle \longmapsto W(|\lambda\rangle) \longmapsto \operatorname{tr}\Big(\rho\ \widehat{\mathbb{E}}_W(0,T)(B)\Big) \in \mathbb{C}. \tag{7.15}$$

There is, however, a freedom in choosing different functions $c$ and $U$. Physically $c$ would correspond to a "rescaling" and $U$ would implement unitary rotations of the dilations spaces w.r.t. other. Since both of those notions are physically unobservable we see $c$ and $U$ as pure gauge related relics.

Since the Weyl operators $W$ are (up to an imaginary factor) generated by the quantum fields, they can naturally be seen as some kind of translation operators. The test function $|\lambda\rangle$ is then, as usual for Wightman quantum fields, the object the quantum field is "smeared out" with. Since $\widehat{\mathbb{E}}_W(0,T)$ is strongly continuous, the cMPS (as the expectation value w.r.t. the state $\rho$), is guaranteed to be, at least, weak-* continuous. This continuity is commonly postulated for physically observable quantities.

## 7.2 Outlook

Since the object $\widehat{\mathbb{E}}_W$ is quite abstract, the definition of cMPS might seem to be impractical. There is, however, a crucial fact one can exploit about this evolution system, i.e. its Cauchy equation. This outlook is designed to state this Cauchy equation, solve it for a trivial case therefore obtain an explicit cMPS, and discuss further solution techniques for this task.

## Weyl Perturbed Lindblad equations

The term $\mathrm{tr}\left(\rho\, \widehat{\mathbb{E}}_W(0,T)(B)\right)$ describes the complete dynamics of our quantum state $\rho$, as modeled by a cMPS. The analysis of this term displays, nevertheless, a challenge. The following theorem (see [Neu15][Lemma 8.6.]) helps approaching this task.

**Theorem 7.6.** *Let $W \in \mathcal{B}(\mathcal{K}_{[0,T]})$ be a Weyl operator, i.e. an operator written as*

$$J_\Theta^\dagger W J_\Theta \overset{t.r.o.}{=} \bigotimes_{i\in I(\Theta)} \begin{pmatrix} 1+\tau_i\,\overline{c}_i & \sqrt{\tau_i}\left\langle\overline{\lambda}_i\right| \\ \sqrt{\tau_i}\left|\overline{\mu}_i\right\rangle & \mathbb{1}_{\mathcal{K}}+\overline{U}_i \end{pmatrix} \in \mathcal{B}(\mathcal{K}_\Theta), \tag{7.16}$$

*with functions $c \in L^1([0,T],\mathbb{C})$, $|\lambda\rangle\,,|\mu\rangle \in L^2([0,T],\mathcal{K})$ and $U \in L^1\left([0,T],\mathcal{B}\left(\mathcal{K}\right)\right)$.*

*Let $0 \le s \le t \le T$. Under the "unbounded assumptions" let $\widehat{\mathbb{E}}(s,t) : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ the evolution system of a cMPS $\omega$ as defined in equation (7.14) via*

$$\widehat{\mathbb{E}}(s,t)(X) := \mathbb{E}_{[s,t]}\left(\mathbb{1}_{\mathcal{B}(\mathcal{K}_{[s,t]})}\otimes X\right)$$

*with $\mathbb{E}_{[0,T]} : \mathcal{B}(\mathcal{K}_{[0,T]}\otimes\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ solving the Cauchy equations*

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\langle\psi\middle|\,\widehat{\mathbb{E}}(s,t)(B)\varphi\right\rangle = \left\langle\psi\middle|\mathcal{L}(s)\widehat{\mathbb{E}}(s,t)(B)\varphi\right\rangle \tag{7.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\psi\middle|\,\widehat{\mathbb{E}}(s,t)(B)\varphi\right\rangle = -\left\langle\psi\middle|\widehat{\mathbb{E}}(s,t)\mathcal{L}(t)(B)\varphi\right\rangle \tag{7.18}$$

*with the generator $\mathcal{L}(t) : \mathcal{B}(\mathcal{H}) \supset dom(\mathcal{L}(t)) \longrightarrow \mathcal{B}(\mathcal{H})$ defined as*

$$\langle\psi|\mathcal{L}(s)(B)\varphi\rangle = \langle K(s)\psi|B\varphi\rangle + \langle\psi|BK(s)\varphi\rangle + \langle L(s)\psi|(\mathbb{1}_{\mathcal{K}}\otimes B)\,L(s)\varphi\rangle\,. \tag{7.19}$$

*Then $\widehat{\mathbb{E}}_W(s,t)(B) := \mathbb{E}_{[s,t]}(W\otimes B)$ defines a weak-$*$ continuous evolution system, solving the Cauchy equation*

$$\frac{\partial}{\partial t}\left\langle\psi\middle|\widehat{\mathbb{E}}_W(s,t)(B)\varphi\right\rangle = -\left\langle\psi\middle|\widehat{\mathbb{E}}_W(s,t)\mathcal{L}_{pert.}(t)(B)\varphi\right\rangle \tag{7.20}$$

*with the "perturbed" Lindbladian*

$$\begin{aligned}\frac{\partial^+}{\partial s}\left\langle\psi\middle|\,\widehat{\mathbb{E}}_W(s,t)(B)\varphi\right\rangle\Big|_{s=t} &= \langle\psi|\mathcal{L}_{pert.}(t)(B)\varphi\rangle \\ &= \langle\psi|(\mathcal{L}(t)+c(t)\mathbb{1}_{\mathcal{B}(\mathcal{H})})(B)\varphi\rangle \\ &\quad+\langle L(t)\psi|(U(t)\otimes B)\,L(t)\varphi\rangle \\ &\quad+\langle\lambda_1(t)\otimes\psi|(\mathbb{1}_{\mathcal{K}}\otimes B)\,L(t)\varphi\rangle \\ &\quad+\langle L(t)\psi|(\mathbb{1}_{\mathcal{K}}\otimes B)\,(\lambda_2(t)\otimes\varphi)\rangle \end{aligned} \tag{7.21}$$

*with* $|\psi\rangle\in\mathcal{D}\subseteq\mathcal{H}$, $B\in\mathcal{B}(\mathcal{H})$, $L(t):\mathcal{D}\longrightarrow\mathcal{K}\otimes\mathcal{H}$.

Via theorem 7.6 we are now able to gain information about the specific form of $\widehat{\mathbb{E}}_W(s,t)(B)$ using the equations (7.20) and (7.21). It states, that the interaction of the quantum system with the cMPS results in a modification of the "unperturbed" generator $\mathcal{L}(t)$. We continue by explicitly solving this PDE for an especially easy example.

## Example: Gaussian cMPS

The equation (7.20) connects the "usual" time evolution of the quantum system generated by $\mathcal{L}(t)$ with the generator $\mathcal{L}_{\text{pert.}}(t)$ arising from interaction with the quantum field. It therefore seems to be a fruitful idea to start with a trivial Lindbladian, i.e. $\mathcal{L}(t)=L(t)=K(t)=0$, and therefore look at a pure gauge process of the form

$$\begin{aligned}\mathcal{L}(t):\,\mathcal{B}(\mathbb{C})\cong\mathbb{C}&\longrightarrow\mathcal{B}(\mathbb{C})\cong\mathbb{C}\\ \mathcal{L}(t)(z)=0=-\sum_\alpha\|\phi_\alpha(t)\|^2 z&+\sum_\alpha\overline{\phi_\alpha(t)}\,z\,\phi_\alpha(t),\end{aligned}\tag{7.22}$$

where $\mathcal{H}=\mathbb{C}$ and $\mathcal{K}=\mathbb{C}^n$ for some $n\in\mathbb{N}$ and $[0,T]\ni t\longmapsto\phi_\alpha(t)\in\mathbb{C}$ are continuously differentiable gauge functions for all $1\leq\alpha\leq n$. We write the bold letter $\boldsymbol{\phi}(t)\in\mathbb{C}^n$ for the n-dimensional vector.
Since this Lindbladian obviously corresponds to a norm continuous evolution system, we analyze the bounded assumptions. Let us recall that there must exist two functions $\widetilde{K}(t):\mathbb{C}\longrightarrow\mathbb{C}$ and $\widetilde{L}(t):\mathbb{C}\longrightarrow\mathbb{C}^n\otimes\mathbb{C}\cong\mathbb{C}^n$ with

B.I. The function $t \longmapsto \widetilde{K}(t)$ is continuous and $\|K(t)\| \le C$ for $t \in [0,T]$.

B.II. The function $t \longmapsto \widetilde{L}(t)$ is continuous.

B.III. For every $t \in [0,T]$ we have that

$$\widetilde{K}^\dagger(t) + \widetilde{K}(t) + \widetilde{L}(t)^\dagger \widetilde{L}(t) \le 0. \tag{7.23}$$

In this example we can easily read of the exact form of these two operators as

$$\widetilde{K}(t)(z) := -\frac{1}{2} \sum_\alpha \|\phi_\alpha(t)\|^2 z \tag{7.24}$$

$$\widetilde{L}(t)(z) := (\phi_1(t)z \ \cdots \ \phi_n(t)z)^T \otimes \mathbb{1}_\mathbb{C}, \tag{7.25}$$

which is exactly the gauge freedom obtained in definition 2.27 with vanishing functions $x(t)$ and $U(t)$. The adjoint operator of $\widetilde{K}(t)$ is obviously $\widetilde{K}(t)$ itself, since $\sum_\alpha \|\phi_\alpha(t)\|^2$ is a real number. The adjoint of $\widetilde{L}(t)$ can be calculated by its action o arbitrary elements $y$ and $v$ as

$$\begin{aligned}
\left\langle \widetilde{L}(t)(z) \middle| y \otimes v \right\rangle &= \left\langle \begin{pmatrix} \phi_1(t)z \\ \vdots \\ \phi_n(t)z \end{pmatrix} \otimes \mathbb{1}_\mathbb{C} \middle| \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \otimes v \right\rangle \\
&= \sum_{\alpha=1}^n \overline{\phi_\alpha(t)\, z}\, y_\alpha \langle \mathbb{1}_\mathbb{C} | v \rangle = \overline{z} \sum_{\alpha=1}^n \overline{\phi_\alpha(t)}\, y_\alpha \langle \mathbb{1}_\mathbb{C} | v \rangle \\
&= \left\langle z \middle| \sum_{\alpha=1}^n \overline{\phi_\alpha(t)}\, y_\alpha \langle \mathbb{1}_\mathbb{C} | v \rangle \right\rangle =: \left\langle z \middle| \widetilde{L}(t)^\dagger (y \otimes v) \right\rangle.
\end{aligned}$$

The three bounded assumptions hold, since

I. $t \longmapsto \widetilde{K}(t)$ is continuous and $\left\| \widetilde{K}(t) \right\| \le \frac{1}{2} \sum_{\alpha=1}^n \sup_{t \in [0,T]} \phi_\alpha(t) < \infty$.

II. $t \longmapsto \widetilde{L}(t)$ is continuous, since the $\phi_\alpha(t)$ are assumed to be continuous.

III. $\widetilde{K}^\dagger(t) + \widetilde{K}(t) + \widetilde{L}(t)^\dagger \widetilde{L}(t) = -\sum_\alpha \|\phi_\alpha(t)\|^2 + \sum_\alpha \|\phi_\alpha(t)\|^2 = 0$.

This gauge freedom results in a perturbed evolution (see theorem 7.6), given by the Lindbladian

$$\begin{aligned}\left\langle\psi\middle|\mathcal{L}_{\text{pert.}}(t)(z)\psi\right\rangle &= \overbrace{\langle\psi|\mathcal{L}(t)(z)\psi\rangle + \langle\psi|z\,c(t)\,\psi\rangle + \left\langle\psi\middle|\widetilde{L}(t)^{\dagger}(U(t)\otimes z)\widetilde{L}(t)\psi\right\rangle}^{=0,\ \text{since}\ \mathcal{L}(t)=c(t)=U(t)=0} \\ &\quad + \left\langle\boldsymbol{\phi}(t)\otimes\psi\,\middle|\,(\mathbb{1}\otimes z)\widetilde{L}(t)\psi\right\rangle + \left\langle\widetilde{L}(t)\psi\,\middle|\,(\boldsymbol{\phi}(t)\otimes z)\psi\right\rangle \\ &= z\left(\|\boldsymbol{\phi}(t)\|^2\langle\psi|\psi\rangle + \|\boldsymbol{\phi}(t)\|^2\langle\psi|\psi\rangle\right) = 2z\|\boldsymbol{\phi}(t)\|^2\langle\psi|\psi\rangle\,.\end{aligned}$$

We are interested in the normalized cMPS $\omega(W) = \operatorname{tr}\left(\rho\,\widehat{\mathbb{E}}_W(s,t)(z)\right)$. We know that the evolution system of interest, i.e. $\widehat{\mathbb{E}}_W(s,t)$, obeys the Cauchy equation

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle\psi\middle|\widehat{\mathbb{E}}_W(s,t)(z)\varphi\right\rangle &= -\left\langle\psi\middle|\widehat{\mathbb{E}}_W(s,t)\mathcal{L}_{\text{pert.}}(t)(z)\varphi\right\rangle \\ &= -2\|\boldsymbol{\phi}(t)\|^2\left\langle\psi\middle|\widehat{\mathbb{E}}_W(s,t)(z)\varphi\right\rangle.\end{aligned}$$

This is a first order PDE and can easily be solved by

$$\widehat{\mathbb{E}}_W(s,t)(z) = \exp\left(-2\int_s^t \left\|\boldsymbol{\phi}(t')\right\|^2 \mathrm{d}t'\right)z. \tag{7.26}$$

So the cMPS acts as the multiplication with the exponential above. Hence the cMPS corresponding to the evolution of the quantum state $\rho \in \mathfrak{T}(\mathcal{H})$ and boundary condition $z \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned}\omega(W) = \operatorname{tr}\left(\rho\,\widehat{\mathbb{E}}_W(s,t)(z)\right) &= \operatorname{tr}\left(\rho\,\exp\left(-2\int_s^t \left\|\boldsymbol{\phi}(t')\right\|^2\mathrm{d}t'\right)z\right) \\ &= \exp\left(-2\int_s^t \left\|\boldsymbol{\phi}(t')\right\|^2\mathrm{d}t'\right)\operatorname{tr}(\rho z) = \exp\left(-2\int_s^t \left\|\boldsymbol{\phi}(t')\right\|^2\mathrm{d}t'\right)z \end{aligned}\tag{7.27}$$

is a Gaussian state. One therefore sees that working with the “wrong” gauge adds Gaussian noise to our system.

## Different Approaches to Obtain Moment Measures

Dealing with non trivial systems, the Cauchy equation could become, in theory, arbitrarily wild. However, one is often not interested in the whole counting statistics, but in the first probability moments.

It is known that the existence of the n-th moment measure is equivalent to the characteristic function being n times differentiable. Let us look at a cMPS with boundary condition $B$ and a Weyl operator depending on a function $|\lambda\rangle \in L^2([0,T],\mathcal{K})$, i.e.

$$W(|\lambda\rangle) \longmapsto \operatorname{tr}\Big(\rho\, \widehat{\mathbb{E}}_{W(|\lambda\rangle)}(0,T)(B)\Big).$$

Since the first moment of the quantum field is given by the first derivative of the characteristic function evaluated at zero it seems to be a fruitful idea to calculate the expression

$$\frac{\partial}{\partial t} \operatorname{tr}\Big(\rho\, \widehat{\mathbb{E}}_{W(t|\lambda\rangle)}(0,T)(B)\Big)\Big|_{t=0}.$$

To gain further information about the evolution system, one could rewrite it using the approximation

$$\widehat{\mathbb{E}}_W(0,T)(B) = \mathcal{T}\exp\left(\int_0^T \mathcal{L}_{\text{pert.}}(t)(B)\,\mathrm{d}t\right), \tag{7.28}$$

where this integral is meant as a continuous version of

$$\widehat{\mathbb{E}}_W(0,T)(B) = \prod_{i\in I(\Theta)} \exp\big(\tau_i \mathcal{L}_{\text{pert.}}(t_i)(B)\big).$$

Or, one could try to solve the Cauchy equation

$$\frac{\partial}{\partial t}\left\langle \psi \,\middle|\, \widehat{\mathbb{E}}_W(s,t)(B)\varphi \right\rangle = -\left\langle \psi \,\middle|\, \widehat{\mathbb{E}}_W(s,t)\mathcal{L}_{\text{pert.}}(t)(B)\varphi \right\rangle,$$

using a variety of tools commonly used for PDEs, such as variational methods.

Since most beam-type measurements can be modeled to be completely random, in the sense that “neighboring” chunks of click events do not interact and do not depend on the corresponding time-step, one could make the ansatz of $C(f)$ being of Poisson form. This choice could simplify the Cauchy equation a lot.

## Further Research Topics

Since we mainly focused on bosonic systems it might be interesting to see whether the fermionic approach would yield more constraints on our cMPS and its defining objects. It would also be interesting to further elaborate the correspondence between the cMPS as outlined here and its first definition from 2010.

Appendix B also presents a specific representation of the Weyl CCR on the phase space, as commonly used in the “quantum harmonic analysis”, describing hybrid systems. The connection of this topic, especially involving its symmetries, seems to be promising.

# Appendix A

# Outsourced Proofs

Due to the long formulas, the page geometry is changed for this appendix to landscape.

## A.1 Proof of Theorem 2.38

*Proof.* The first two identities are trivial. To prove the third identity we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t\xi)W(\eta)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}e^{itB_\pi(\xi)}W(\eta)\Big|_{t=0} = iB_\pi(\xi)W(t\xi)W(\eta)\Big|_{t=0} = iB_\pi(\xi)W(\eta).$$

Using the Weyl CCR we also get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} W(t\xi) W(\eta) \Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t} e^{i\sigma(t\xi,\eta)/2}\, W(t\xi + \eta) \Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} e^{i\sigma(t\xi,\eta)}\, W(\eta) W(t\xi) \Big|_{t=0} \\
&= i\sigma(\xi,\eta) e^{it\sigma(\xi,\eta)}\, W(\eta) W(t\xi) \Big|_{t=0} + e^{it\sigma(\xi,\eta)}\, W(\eta) i B_\pi(\xi) W(t\xi) \Big|_{t=0} \\
&= i\sigma(\xi,\eta) W(\eta) + i W(\eta) B_\pi(\xi).
\end{aligned}
$$

Hence we see

$$
B_\pi(\xi) W(\eta) = \sigma(\xi,\eta) W(\eta) + W(\eta) B_\pi(\xi) \quad \Longleftrightarrow \quad [B_\pi(\xi), W(\eta)] = \sigma(\xi,\eta) W(\eta).
$$

The last commutation relation is then done by differentiating w.r.t. two parameters in the same way. More formally

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s} W(t\xi) W(s\eta) \Big|_{t=s=0} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s} e^{itB_\pi(\xi)} e^{isB_\pi(\eta)} \Big|_{t=s=0} = iB_\pi(\xi) e^{itB_\pi(\xi)} iB_\pi(\eta) e^{isB_\pi(\eta)} \Big|_{t=s=0} = -B_\pi(\xi) B_\pi(\eta).
$$

And furthermore

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s} W(t\xi) W(s\eta) \Big|_{t=s=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s} e^{its\,\sigma(\xi,\eta)}\, W(s\eta) W(t\xi) \Big|_{t=s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \Big( it\sigma(\xi,\eta)\, e^{its\,\sigma(\xi,\eta)}\, W(s\eta) W(t\xi) + e^{its\,\sigma(\xi,\eta)}\, iB_\pi(\eta) W(s\eta) W(t\xi) \Big) \Big|_{t=s=0}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(it\sigma(\xi,\eta)W(t\xi) + iB_\pi(\eta)W(t\xi)\right)\Big|_{t=0} \\
&= \left(i\sigma(\xi,\eta)W(t\xi) + it\sigma(\xi,\eta)iB_\pi(\xi)W(t\xi) + iB_\pi(\eta)iB_\pi(\xi)W(t\xi)\right)\Big|_{t=0} \\
&= i\sigma(\xi,\eta) - B_\pi(\eta)B_\pi(\xi).
\end{aligned}
$$

So finally one has

$$
-B_\pi(\xi)B_\pi(\eta) = i\sigma(\xi,\eta) - B_\pi(\eta)B_\pi(\xi) \iff B_\pi(\eta)B_\pi(\xi) - B_\pi(\xi)B_\pi(\eta) = i\sigma(\xi,\eta) \iff [B_\pi(\xi), B_\pi(\eta)] = -i\sigma(\xi,\eta).
$$

□

## A.2 Proof of Theorem 5.1

*Proof.* Since we have

$$
\bigotimes_{i\in I(\Theta)} \widetilde{W}_i = \bigotimes_{i\in I(\Theta)} J_i^\dagger \left( \bigotimes_{j\in I(\Xi|_i)} W_j \right) J_i,
$$

it is clear that we can restrict our analysis to the subinterval $\mathcal{K}_{\Xi|_i}$. Calculating the upper left matrix elements of $\widetilde{W}_i$ is a straight forward calculation

$$\left\langle J_i \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \middle| \left( \bigotimes_{j\in I(\Xi|_i)} W_j \right) J_i \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \right\rangle = \left\langle \bigotimes_{k\in I(\Xi|_i)} \begin{pmatrix} |0\rangle \\ 0\end{pmatrix} \middle| \left(\bigotimes_{j\in I(\Xi|_i)} W_j\right) \left(\bigotimes_{l\in I(\Xi|_i)} \begin{pmatrix}|0\rangle\\0\end{pmatrix}\right)\right\rangle$$

$$= \left\langle \bigotimes_{k\in I(\Xi|_i)} \begin{pmatrix}|0\rangle\\0\end{pmatrix} \middle| \bigotimes_{j\in I(\Xi|_i)} \begin{pmatrix} c_j\,|0\rangle \\ \sqrt{\tau_j}\,|\lambda_j\rangle\,|0\rangle\end{pmatrix}\right\rangle = \prod_{j\in I(\Xi|_i)} \left\langle \begin{pmatrix}|0\rangle\\0\end{pmatrix}\middle|\begin{pmatrix}c_j\,|0\rangle\\ \sqrt{\tau_j}\,|\lambda_j\rangle\,|0\rangle\end{pmatrix}\right\rangle = \prod_{j\in I(\Xi|_i)} c_j.$$

The upper right and lower left matrix elements work completely analogously, here shown for the $\langle\nu|$-term

$$\left\langle J_i \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \middle| \left( \bigotimes_{j\in I(\Xi|_i)} W_j \right) J_i \begin{pmatrix} 0 \\ |\alpha\rangle \end{pmatrix} \right\rangle = \left\langle \bigotimes_{k\in I(\Xi|_i)} \begin{pmatrix} |0\rangle \\ 0\end{pmatrix} \middle| \left(\bigotimes_{j\in I(\Xi|_i)} W_j\right) \sum_{l\in I(\Xi|_i)} \sqrt{\frac{\tau_l}{\tau_i}}\,|\alpha@l\rangle \right\rangle$$

$$= \sum_{l\in I(\Xi|_i)} \sqrt{\frac{\tau_l}{\tau_i}} \left\langle \bigotimes_{k\in I(\Xi|_i)} \begin{pmatrix}|0\rangle\\0\end{pmatrix} \middle| \bigotimes_{j\neq l\in I(\Xi|_i)} \begin{pmatrix} c_j\,|0\rangle \\ \sqrt{\tau_j}\,|\lambda_j\rangle\,|0\rangle\end{pmatrix} \otimes \begin{pmatrix} \frac{c_l}{c_l}\sqrt{\tau_l}\,\langle\nu_l|\alpha\rangle \\ O_l\,|\alpha\rangle + \tau_l \frac{|\lambda_l\rangle\langle\nu_l|}{c_l}\,|\alpha\rangle \end{pmatrix}\right\rangle$$

$$= \left(\prod_{j\in I(\Xi|_i)} c_j\right) \sum_{l\in I(\Xi|_i)} \langle 0|\, \frac{\tau_l}{c_l\sqrt{\tau_i}}\,\langle\nu_l|\alpha\rangle = \langle 0| \left(\prod_{j\in I(\Xi|_i)} c_j\right) \sqrt{\tau_i}\,\mathfrak{M}_i^{\Xi}\left(\frac{\langle\nu|}{c}\right)|\alpha\rangle\,.$$

The lower right matrix elements require a bit more thought, because having two (possibly distinct) events leads us to two different contributions

$$
\begin{aligned}
&\left\langle J_i \begin{pmatrix} 0 \\ |\alpha\rangle \end{pmatrix} \middle| \left( \bigotimes_{j\in I(\Xi|_i)} W_j \right) J_i \begin{pmatrix} 0 \\ |\beta\rangle \end{pmatrix} \right\rangle \\
&= \left\langle \sum_{l\in I(\Xi|_i)} \sqrt{\frac{\tau_l}{\tau_i}}\, |\alpha@l\rangle \middle| \sum_{m\in I(\Xi|_i)} \sqrt{\frac{\tau_m}{\tau_i}} \bigotimes_{j\in I(\Xi|_i)} \begin{pmatrix} c_j & \sqrt{\tau_j}\,\langle\nu_j| \\ \sqrt{\tau_j}\,|\lambda_j\rangle & O_j + \tau_j \frac{|\lambda_j\rangle\langle\nu_j|}{c_j} \end{pmatrix} |\beta@m\rangle \right\rangle \\
&= \sum_{l,m\in I(\Xi|_i)} \frac{\sqrt{\tau_l\tau_m}}{\tau_i} \left\langle \alpha@l \middle| \bigotimes_{j\neq m\in I(\Xi|_i)} \begin{pmatrix} c_j\,|0\rangle \\ \sqrt{\tau_j}\,|\lambda_j\rangle\,|0\rangle \end{pmatrix} \otimes \begin{pmatrix} \sqrt{\tau_m}\,\langle\nu_m|\beta\rangle \\ \left(O_m + \tau_m \frac{|\lambda_m\rangle\langle\nu_m|}{c_m}\right)|\beta\rangle \end{pmatrix} \right\rangle \\
&= \sum_{l,m\in I(\Xi|_i)} \frac{\sqrt{\tau_l\,\tau_m}}{\tau_i} \left[ \delta_{lm} \prod_{j\neq l\in I(\Xi|_i)} c_j \left( \frac{c_l}{c_l}\,\langle\alpha|O_l|\beta\rangle + \frac{c_l}{c_l^2}\tau_l\,\langle\alpha|\lambda_l\rangle\,\langle\nu_l|\beta\rangle \right) + (1-\delta_{lm}) \left( \prod_{j\neq l\neq m\in I(\Xi|_i)} c_j \right) \sqrt{\tau_l\,\tau_m}\,\frac{c_l\,c_m}{c_l\,c_m}\,\langle\alpha|\lambda_l\rangle\,\langle\nu_m|\beta\rangle \right] \\
&= \left( \prod_{j\in I(\Xi|_i)} c_j \right) \sum_{l,m\in I(\Xi|_i)} \frac{\sqrt{\tau_l\,\tau_m}}{\tau_i} \left[ \delta_{lm} \left( \frac{1}{c_l}\,\langle\alpha|O_l|\beta\rangle + \frac{\tau_l}{c_l^2}\,\langle\alpha|\lambda_l\rangle\,\langle\nu_l|\beta\rangle \right) + (1-\delta_{lm}) \frac{\sqrt{\tau_l\,\tau_m}}{c_l\,c_m}\,\langle\alpha|\lambda_l\rangle\,\langle\nu_m|\beta\rangle \right] \\
&= \left( \prod_{j\in I(\Xi|_i)} c_j \right) \sum_{l,m\in I(\Xi|_i)} \left[ \frac{\tau_l\,\tau_m}{\tau_i} \frac{1}{c_l\,c_m}\,\langle\alpha|\lambda_l\rangle\,\langle\nu_m|\beta\rangle + \frac{\tau_l}{\tau_i} \frac{\langle\alpha|O_l|\beta\rangle}{c_l} \right] \\
&= \left( \prod_{j\in I(\Xi|_i)} c_j \right) \left[ \left\langle \alpha \middle| \tau_i\, \mathfrak{M}_i^{\Xi}\left(\frac{|\lambda\rangle}{c}\right) \mathfrak{M}_i^{\Xi}\left(\frac{\langle\nu|}{c}\right) \middle| \beta \right\rangle + \left\langle \alpha \middle| \mathfrak{M}_i^{\Xi}\left(\frac{O}{c}\right) \middle| \beta \right\rangle \right].
\end{aligned}
$$

Collecting all terms then proves the theorem. □

## A.3 Proof of Theorem 5.8

*Proof.* Analogously to before we use that each $W_\Theta(|\lambda\rangle)$ is unitary, since $\Phi(\sqrt{\tau_i}\,|\overline{\lambda}_i\rangle)$ is clearly skew-hermitian. Since $|\lambda\rangle$ is bounded, it is sufficient to look at the following norm difference

$$\begin{aligned}\left\|W_\Lambda(|\lambda\rangle)J_{\Lambda\Theta}\,|\varphi_\Theta\rangle - J_{\Lambda\Xi}W_\Xi(|\lambda\rangle)J_{\Xi\Theta}\,|\varphi_\Theta\rangle\right\|^2 &= 2\|\varphi_\Theta\|^2 - 2\,\mathrm{Re}\,\langle W_\Lambda(|\lambda\rangle)J_{\Lambda\Theta}\varphi_\Theta | J_{\Lambda\Xi}W_\Xi(|\lambda\rangle)J_{\Xi\Theta}\varphi_\Theta\rangle \\ &= 2\|\varphi_\Theta\|^2 - 2\,\mathrm{Re}\left\langle J_{\Xi\Theta}\varphi_\Theta \middle| \left(J^\dagger_{\Lambda\Xi}W^\dagger_\Lambda(|\lambda\rangle)J_{\Lambda\Xi}W_\Xi(|\lambda\rangle)\right) J_{\Xi\Theta}\varphi_\Theta\right\rangle.\end{aligned}$$

We therefore see that strong convergence of $W_\Theta(|\lambda\rangle)$ is equivalent to the weak-* convergence of $J^\dagger_{\Lambda\Xi}W^\dagger_\Lambda(|\lambda\rangle)J_{\Lambda\Xi}W_\Xi(|\lambda\rangle)$ to the identity. We use theorem 5.1 to calculate that the operator

$$W^\dagger_\Lambda(|\lambda\rangle) = \bigotimes_{k\in I(\Lambda)} \begin{pmatrix} \cos(\theta_k) & -\sin(\theta_k)\,\langle e_{\lambda_k}| \\ \sin(\theta_k)\,|e_k\rangle & \mathbb{1} + (\cos(\theta_k)-1)\,|e_k\rangle\langle e_k| \end{pmatrix},$$

is embedded of the form

$$J^\dagger_{\Lambda\Xi}W^\dagger_\Lambda(|\lambda\rangle)J_{\Lambda\Xi} = \bigotimes_{i\in I(\Xi)} \left(\prod_{j\in I(\Lambda|_i)} \cos(\theta_j)\right) \begin{pmatrix} 1 & -\sum_{l\in I(\Lambda|_i)} \sqrt{\frac{\tau_l}{\tau_i}}\tan(\theta_l)\,\langle e_l| \\ \sum_{l\in I(\Lambda|_i)} \sqrt{\frac{\tau_l}{\tau_i}}\tan(\theta_l)\,|e_l\rangle & \sum_{l\in I(\Lambda|_i)} \frac{\tau_l}{\tau_i}\frac{1}{\cos(\theta_l)}\mathbb{1} + A_i \end{pmatrix},$$

with a rather long (but later neglected) term

$$A_i = \sum_{l\in I(\Lambda|_i)} \frac{\tau_l}{\tau_i}\frac{\cos(\theta_l)-1}{\cos(\theta_l)}\,|e_l\rangle\!\langle e_l| - \sum_{\substack{l,k\in I(\Lambda|_i)\\ l\neq k}} \frac{\sqrt{\tau_k\,\tau_l}}{\tau_i}\tan(\theta_l)\tan(\theta_k)\,|e_k\rangle\!\langle e_l| \in \mathcal{B}(\mathcal{K}).$$

The strategy from here on is straightforward: We calculate the matrix multiplication of this operator with another $W_\Xi(|\lambda\rangle)$ and wright down the upcoming matrix elements in orders of $\tau$. The full operator has the form

$$J^\dagger_{\Lambda\Xi}W^\dagger_\Lambda(|\lambda\rangle)J_{\Lambda\Xi}W_\Xi(|\lambda\rangle) = \bigotimes_{i\in I(\Xi)}\left(\prod_{j\in I(\Lambda|_i)}\cos(\theta_j)\right)\begin{pmatrix} W_{\mathrm{UL}} & W_{\mathrm{UR}} \\ W_{\mathrm{LL}} & W_{\mathrm{LR}}\end{pmatrix},$$

using the abbreviations

$$\begin{aligned}
W_{\mathrm{UL}} &= \cos(\theta_i) + \sin(\theta_i)\sum_{l\in I(\Lambda|_i)}\sqrt{\frac{\tau_l}{\tau_i}}\tan(\theta_l)\,\langle e_l|e_i\rangle \\
W_{\mathrm{UR}} &= \sin(\theta_i)\,\langle e_i| - \sum_{l\in I(\Lambda|_i)}\sqrt{\frac{\tau_l}{\tau_i}}\tan(\theta_l)\,\langle e_l| - (\cos(\theta_i)-1)\sum_{l\in I(\Lambda|_i)}\sqrt{\frac{\tau_l}{\tau_i}}\tan(\theta_l)\,\langle e_l|e_i\rangle\,\langle e_i| \\
W_{\mathrm{LL}} &= \cos(\theta_i)\sum_{l\in I(\Lambda|_i)}\sqrt{\frac{\tau_l}{\tau_i}}\tan(\theta_l)\,|e_l\rangle - \sin(\theta_i)\sum_{l\in I(\Lambda|_i)}\frac{\tau_l}{\tau_i}\frac{1}{\cos(\theta_l)}\,|e_i\rangle - A_i\sin(\theta_i)\,|e_i\rangle \\
W_{\mathrm{LR}} &= \sin(\theta_i)\sum_{l\in I(\Lambda|_i)}\sqrt{\frac{\tau_l}{\tau_i}}\tan(\theta_l)\,|e_l\rangle\!\langle e_i| + A_i + \sum_{l\in I(\Lambda|_i)}\frac{\tau_l}{\tau_i}\frac{1}{\cos(\theta_l)}\mathbb{1} + \sum_{l\in I(\Lambda|_i)}\frac{\tau_l}{\tau_i}\frac{\cos(\theta_i)-1}{\cos(\theta_l)}\,|e_i\rangle\!\langle e_l| + A_i(\cos(\theta_i)-1)\,|e_i\rangle\!\langle e_i|\,.
\end{aligned}$$

To prove that this expression converges to the identity, we need to rewrite every term in order to see whether terms cancel or not and approximate the non vanishing terms to relevant order in $\tau_i$.

<u>Order approximation and common refinement:</u>

Before we are able to approximate the full expression correctly, we need to restrict $|\lambda\rangle$ to step functions again, i.e. we assume $|\lambda\rangle \in P_\Xi(L^2([0,T],\mathcal{K}))$, since $\Xi$ is the coarsest interval decomposition involved. Note that $i \in I(\Xi)$ and $l \in I(\Lambda|_i)$ with $l \neq i$ and because $\Lambda|_i \subseteq \Xi$ we have $\left\|\overline{\lambda}_l\right\| = \left\|\overline{\lambda}_i\right\|$, which implies $|e_i\rangle = |e_l\rangle$ and also $\sum\limits_{l\in I(\Lambda|_i)} \tau_l = \tau_i$.

The trigonometric functions are to first orders given by

$$\sin\!\left(\sqrt{\tau_l}\,\left\|\overline{\lambda}_l\right\|\right) = \sqrt{\tau_l}\,\left\|\overline{\lambda}_l\right\| + \mathcal{O}\left(\tau_l^{3/2}\right), \qquad \cos\!\left(\sqrt{\tau_l}\,\left\|\overline{\lambda}_l\right\|\right) = 1 - \frac{\tau_l\left\|\overline{\lambda}_l\right\|^2}{2} + \mathcal{O}\left(\tau_l^2\right), \qquad \tan\!\left(\sqrt{\tau_l}\,\left\|\overline{\lambda}_l\right\|\right) = \sqrt{\tau_l}\,\left\|\overline{\lambda}_l\right\| + \mathcal{O}\left(\tau_l^{3/2}\right).$$

Since the full matrix is quite massive, we split the analysis into the smaller matrix blocks and apply those approximations separately.

<u>$\mathbb{C}$-valued part $W_{\mathrm{UL}}$:</u>

In this matrix element only terms of zeroth and first order of $\tau_i$ are allowed. We calculate:

$$\begin{aligned} W_{\mathrm{UL}} &= \cos(\theta_i) + \sin(\theta_i) \sum_{l\in I(\Lambda|_i)} \sqrt{\frac{\tau_l}{\tau_i}}\tan(\theta_l)\,\langle e_l|e_i\rangle \overset{\text{t.r.o.}}{=} 1 - \frac{\tau_i\left\|\overline{\lambda}_i\right\|^2}{2} + \sqrt{\tau_i}\left\|\overline{\lambda}_i\right\| \sum_{l\in I(\Lambda|_i)} \frac{\tau_l}{\sqrt{\tau_i}}\left\|\overline{\lambda}_l\right\| \overbrace{\langle e_i|e_i\rangle}^{=1} \\ &= 1 - \frac{\tau_i\left\|\overline{\lambda}_i\right\|^2}{2} + \tau_i\left\|\overline{\lambda}_l\right\|^2 = 1 + \frac{\tau_i\left\|\overline{\lambda}_i\right\|^2}{2}. \end{aligned}$$

$\mathcal{K}$-valued parts $W_{\text{UR}}$, $W_{\text{LL}}$:

Both $\mathcal{K}$-valued terms are zero as we will see now:

$$
\begin{aligned}
W_{\text{UR}} &= \sin(\theta_i)\langle e_i| - \sum_{l\in I(\Lambda|_i)} \sqrt{\frac{\tau_l}{\tau_i}}\tan(\theta_l)\langle e_l| - (\cos(\theta_i)-1)\sum_{l\in I(\Lambda|_i)} \sqrt{\frac{\tau_l}{\tau_i}}\tan(\theta_l)\overbrace{\langle e_l|e_i\rangle}^{=1}\langle e_i| \\
&\overset{\text{t.r.o.}}{=} \sqrt{\tau_i}\,\|\overline{\lambda}_i\|\,\langle e_i| - \sqrt{\tau_i}\sum_{l\in I(\Lambda|_i)}\frac{\tau_l}{\tau_i}\|\overline{\lambda}_l\|\,\langle e_l| + \tau_i^{3/2}\frac{\|\overline{\lambda}_i\|^2}{2}\sum_{l\in I(\Lambda|_i)}\frac{\tau_l}{\tau_i}\|\overline{\lambda}_l\|\,\langle e_i| \\
&= \underbrace{\sqrt{\tau_i}\,\|\overline{\lambda}_i\|\,\langle e_i| - \sqrt{\tau_i}\,\|\overline{\lambda}_i\|\,\langle e_i|}_{=0} + \mathcal{O}(\tau_i^{3/2}).
\end{aligned}
$$

And respectively

$$
\begin{aligned}
W_{\text{LL}} &= \cos(\theta_i)\sum_{l\in I(\Lambda|_i)}\sqrt{\frac{\tau_l}{\tau_i}}\tan(\theta_l)\,|e_l\rangle - \sin(\theta_i)\sum_{l\in I(\Lambda|_i)}\frac{\tau_l}{\tau_i}\frac{1}{\cos(\theta_l)}\,|e_i\rangle - A_i\sin(\theta_i)\,|e_i\rangle \\
&= \Bigg[\cos(\theta_i)\sum_{l\in I(\Lambda|_i)}\sqrt{\frac{\tau_l}{\tau_i}}\tan(\theta_l)\underbrace{-\sin(\theta_i)\sum_{l\in I(\Lambda|_i)}\frac{\tau_l}{\tau_i}\frac{1}{\cos(\theta_l)} + \sin(\theta_i)\sum_{l\in I(\Lambda|_i)}\frac{\tau_l}{\tau_i}\frac{1}{\cos(\theta_l)}}_{=0} - \sin(\theta_i) \\
&\qquad - \sin(\theta_i)\sum_{\substack{l,k\in I(\Lambda|_i)\\ l\neq k}}\frac{\sqrt{\tau_k\,\tau_l}}{\tau_i}\tan(\theta_l)\tan(\theta_k)\Bigg]\,|e_i\rangle
\end{aligned}
$$

$$\overset{\text{t.r.o.}}{=} \left[ \left( 1 - \frac{\tau_i \left\| \overline{\lambda}_i \right\|^2}{2} \right) \sum_{l \in I(\Lambda|_i)} \frac{\tau_l}{\sqrt{\tau_i}} \left\| \overline{\lambda}_l \right\| - \sqrt{\tau_i} \left\| \overline{\lambda}_i \right\| - \sqrt{\tau_i} \left\| \overline{\lambda}_i \right\| \sum_{\substack{l,k \in I(\Lambda|_i) \\ l \neq k}} \frac{\tau_k \, \tau_l}{\tau_i} \left\| \overline{\lambda}_l \right\| \left\| \overline{\lambda}_k \right\| \right] |e_i\rangle$$

$$= \left[ \underbrace{\sqrt{\tau_i} \left\| \overline{\lambda}_i \right\| - \sqrt{\tau_i} \left\| \overline{\lambda}_i \right\|}_{=0} - \tau_i^{3/2} \frac{\left\| \overline{\lambda}_i \right\|^3}{2} - \tau_i^{3/2} \left\| \overline{\lambda}_i \right\| \sum_{\substack{l,k \in I(\Lambda|_i) \\ l \neq k}} \frac{\tau_k \, \tau_l}{\tau_i \tau_i} \left\| \overline{\lambda}_l \right\| \left\| \overline{\lambda}_k \right\| \right] |e_i\rangle = \mathcal{O}(\tau_i^{3/2}).$$

<u>$\mathcal{B}(\mathcal{K})$-valued part $W_{\mathrm{LR}}$:</u>

$$\begin{aligned}
W_{\mathrm{LR}} &= \sin(\theta_i) \sum_{l \in I(\Lambda|_i)} \sqrt{\frac{\tau_l}{\tau_i}} \tan(\theta_l) \, |e_l\rangle\langle e_i| + A_i + \sum_{l \in I(\Lambda|_i)} \frac{\tau_l}{\tau_i} \frac{\mathbb{1}}{\cos(\theta_l)} + \sum_{l \in I(\Lambda|_i)} \frac{\tau_l}{\tau_i} \frac{\cos(\theta_i) - 1}{\cos(\theta_l)} \, |e_i\rangle\langle e_l| + A_i(\cos(\theta_i) - 1) \, |e_i\rangle\langle e_i| \\
&= \sin(\theta_i) \sum_{l \in I(\Lambda|_i)} \sqrt{\frac{\tau_l}{\tau_i}} \tan(\theta_l) \, |e_i\rangle\langle e_i| + \sum_{l \in I(\Lambda|_i)} \frac{\tau_l}{\tau_i} \frac{\mathbb{1}}{\cos(\theta_l)} + \sum_{l \in I(\Lambda|_i)} \frac{\tau_l}{\tau_i} \frac{\cos(\theta_i)}{\cos(\theta_l)} \, |e_i\rangle\langle e_i| - \sum_{l \in I(\Lambda|_i)} \frac{\tau_l}{\tau_i} \frac{|e_i\rangle\langle e_i|}{\cos(\theta_l)} + A_i \cos(\theta_i) \, |e_i\rangle\langle e_i| \\
&= \sin(\theta_i) \sum_{l \in I(\Lambda|_i)} \sqrt{\frac{\tau_l}{\tau_i}} \tan(\theta_l) \, |e_i\rangle\langle e_i| + \sum_{l \in I(\Lambda|_i)} \frac{\tau_l}{\tau_i} \frac{\mathbb{1}}{\cos(\theta_l)} - \sum_{l \in I(\Lambda|_i)} \frac{\tau_l}{\tau_i} \frac{|e_i\rangle\langle e_i|}{\cos(\theta_l)} \\
&\qquad + \cos(\theta_i) \sum_{l \in I(\Lambda|_i)} \frac{\tau_l}{\tau_i} \, |e_i\rangle\langle e_i| - \cos(\theta_i) \sum_{\substack{l,k \in I(\Lambda|_i) \\ l \neq k}} \frac{\sqrt{\tau_k \, \tau_l}}{\tau_i} \tan(\theta_l) \tan(\theta_k) \, |e_i\rangle\langle e_i| \\
&\overset{\text{t.r.o.}}{=} \sum_{l \in I(\Lambda|_i)} \frac{\tau_l}{\tau_i} \mathbb{1} + \mathcal{O}(\tau_i) = \mathbb{1} + \mathcal{O}(\tau_i).
\end{aligned}$$

Combining all of this we have

$$J^\dagger_{\Lambda\Xi} W^\dagger_\Lambda(|\lambda\rangle) J_{\Lambda\Xi} W_\Xi(|\lambda\rangle) \overset{\text{t.r.o.}}{=} \bigotimes_{i\in I(\Xi)} \left(\prod_{j\in I(\Lambda|_i)} 1 - \frac{\tau_j \left\|\bar{\lambda}_j\right\|^2}{2}\right) \begin{pmatrix} 1+\frac{\tau_i\left\|\bar{\lambda}_i\right\|^2}{2} & 0 \\ 0 & \mathbb{1}\end{pmatrix}.$$

And taking the limit one finally obtains

$$\lim_{\Lambda\gg\Xi} J^\dagger_{\Lambda\Xi} W^\dagger_\Lambda(|\lambda\rangle) J_{\Lambda\Xi} W_\Xi(|\lambda\rangle) = \bigotimes_{i\in I(\Xi)} \exp\left(-\int_{[t_{i-1},t_i)} \frac{\left\|\bar{\lambda}_i\right\|^2}{2}\right) \exp\left(\int_{[t_{i-1},t_i)} \frac{\left\|\bar{\lambda}_i\right\|^2}{2}\right) \mathbb{1} = \mathbb{1},$$

which proves the theorem. □

## A.4 Proof of Theorem 5.11

*Proof.* The explicit calculation of the operator multiplication is, due to the long terms, shifted here. We calculate the desired expression to relevant order:

$$
\begin{aligned}
&W_\Theta(|\lambda\rangle)W_\Theta(|\mu\rangle)W_\Theta(-|\lambda+\mu\rangle) \\
&\overset{\text{t.r.o.}}{=} \bigotimes_{i\in I(\Theta)} \begin{pmatrix} 1-\frac{\tau_i}{2}\left\|\overline{\lambda}_i\right\|^2 & \sqrt{\tau_i}\left\langle\overline{\lambda}_i\right| \\ -\sqrt{\tau_i}\left|\overline{\lambda}_i\right\rangle & \mathbb{1}\end{pmatrix} \begin{pmatrix} 1-\frac{\tau_i}{2}\left\|\overline{\mu}_i\right\|^2 & \sqrt{\tau_i}\left\langle\overline{\mu}_i\right| \\ -\sqrt{\tau_i}\left|\overline{\mu}_i\right\rangle & \mathbb{1}\end{pmatrix} \begin{pmatrix} 1-\frac{\tau_i}{2}\left\|\overline{\lambda}_i+\overline{\mu}_i\right\|^2 & -\sqrt{\tau_i}\left\langle\overline{\lambda}_i+\overline{\mu}_i\right| \\ \sqrt{\tau_i}\left|\overline{\lambda}_i+\overline{\mu}_i\right\rangle & \mathbb{1}\end{pmatrix} \\
&= \bigotimes_{i\in I(\Theta)} \begin{pmatrix} 1-\frac{\tau_i}{2}\left\|\overline{\lambda}_i\right\|^2-\frac{\tau_i}{2}\left\|\overline{\mu}_i\right\|^2-\tau_i\left\langle\overline{\lambda}_i\middle|\overline{\mu}_i\right\rangle & \sqrt{\tau_i}\left\langle\overline{\lambda}_i\right|+\sqrt{\tau_i}\left\langle\overline{\mu}_i\right| \\ -\sqrt{\tau_i}\left|\overline{\lambda}_i\right\rangle-\sqrt{\tau_i}\left|\overline{\mu}_i\right\rangle & \mathbb{1}\end{pmatrix} \begin{pmatrix} 1-\frac{\tau_i}{2}\left\|\overline{\lambda}_i+\overline{\mu}_i\right\|^2 & -\sqrt{\tau_i}\left\langle\overline{\lambda}_i+\overline{\mu}_i\right| \\ \sqrt{\tau_i}\left|\overline{\lambda}_i+\overline{\mu}_i\right\rangle & \mathbb{1}\end{pmatrix} \\
&= \bigotimes_{i\in I(\Theta)} \begin{pmatrix} 1+\tau_i\left[-\frac{1}{2}\left\|\overline{\lambda}_i\right\|^2-\frac{1}{2}\left\|\overline{\mu}_i\right\|^2-\frac{1}{2}\left\|\overline{\lambda}_i+\overline{\mu}_i\right\|^2+\left\langle\overline{\mu}_i\middle|\overline{\lambda}_i\right\rangle+\left\|\overline{\lambda}_i\right\|^2+\left\|\overline{\mu}_i\right\|^2\right] & \sqrt{\tau_i}\left[\left\langle\overline{\lambda}_i\right|+\left\langle\overline{\mu}_i\right|-\left\langle\overline{\lambda}_i+\overline{\mu}_i\right|\right] \\ -\sqrt{\tau_i}\left[\left|\overline{\lambda}_i\right\rangle+\left|\overline{\mu}_i\right\rangle-\left|\overline{\lambda}_i+\overline{\mu}_i\right\rangle\right] & \mathbb{1}\end{pmatrix}.
\end{aligned}
$$

The off-diagonal terms are trivially zero. Looking at the upper left matrix element and using some norm reformulations we see

$$
\begin{aligned}
&1+\tau_i\left[-\frac{1}{2}\left\|\overline{\lambda}_i\right\|^2-\frac{1}{2}\left\|\overline{\mu}_i\right\|^2-\frac{1}{2}\left\|\overline{\lambda}_i+\overline{\mu}_i\right\|^2+\left\langle\overline{\mu}_i\middle|\overline{\lambda}_i\right\rangle+\left\|\overline{\mu}_i+\overline{\lambda}_i\right\|^2-2\operatorname{Re}(\left\langle\overline{\mu}_i\middle|\overline{\lambda}_i\right\rangle)\right] \\
&\quad= 1+\tau_i\left[-\frac{1}{2}\left\|\overline{\lambda}_i\right\|^2-\frac{1}{2}\left\|\overline{\mu}_i\right\|^2+\frac{1}{2}\left\|\overline{\lambda}_i+\overline{\mu}_i\right\|^2+\left\langle\overline{\mu}_i\middle|\overline{\lambda}_i\right\rangle-\left\langle\overline{\mu}_i\middle|\overline{\lambda}_i\right\rangle-\left\langle\overline{\lambda}_i\middle|\overline{\mu}_i\right\rangle\right] \\
&\quad= 1+\tau_i\left[\operatorname{Re}(\left\langle\overline{\mu}_i\middle|\overline{\lambda}_i\right\rangle)-\left\langle\overline{\lambda}_i\middle|\overline{\mu}_i\right\rangle\right] = 1+\tau_i\left[\frac{1}{2}\left\langle\overline{\mu}_i\middle|\overline{\lambda}_i\right\rangle-\frac{1}{2}\left\langle\overline{\lambda}_i\middle|\overline{\mu}_i\right\rangle\right] = 1-i\tau_i\operatorname{Im}(\left\langle\overline{\lambda}_i\middle|\overline{\mu}_i\right\rangle).
\end{aligned}
$$

One therefore has

$$
\begin{aligned}
\varinjlim_{\Theta\in\mathfrak{Z}([0,T])} W_\Theta(|\lambda\rangle)W_\Theta(|\mu\rangle)W_\Theta(-\,|\lambda+\mu\rangle) &\overset{\text{t.r.o.}}{=} \varinjlim_{\Theta\in\mathfrak{Z}([0,T])} \begin{pmatrix} 1 - i\tau_i \operatorname{Im}(\langle\overline{\lambda}_i|\overline{\mu}_i\rangle) & 0 \\ 0 & \mathbb{1} \end{pmatrix} \\
&= \exp\left( \int_0^T -i \operatorname{Im}(\langle\overline{\lambda}|\overline{\mu}\rangle)\mathrm{d}t \right) \mathbb{1} = \exp\bigl(-iT\operatorname{Im}(\langle\overline{\lambda}|\overline{\mu}\rangle)\bigr)\mathbb{1},
\end{aligned}
$$

which was to prove.

□

# Appendix B

# Quantum Harmonic Analysis on Phase Space

We now want to restrict ourselves to the, considerably, most famous example of a symplectic space. Let $(\Xi, \sigma)$ be the phase space of classical mechanics, i.e. a $2N$-dimensional vector space with symplectic form $\sigma$. Furthermore we will consider the following:

i) **Symplectic space**: $\Xi = \operatorname{span}\left\{(p,q) \;\middle|\; p \in \mathbb{R}^N_{\geq 0}, q \in \mathbb{R}^N\right\}$, equipped with the Lebesgue measure $\mathrm{d}^N\xi = (2\pi\hbar)^{-N}\mathrm{d}^N p\,\mathrm{d}^N q$. We set $\hbar = 1$.

ii) **Symplectic form**: $\sigma((p,q),(p',q')) = p \cdot q' - q \cdot p'$.

iii) **Representation space**: $\mathcal{H} = L^2(\mathbb{R}^N, \mathrm{d}^N x)$.

To keep notation simple, we will denote the complex conjugate of the complex number $z$ in this (and only this) chapter by $\overline{z}$. Otherwise, additional stars would make the proofs in this chapter barely readable.

Since we now have defined the phase space with its symplectic form and having a Hilbert space we can construct an explicit representation of the Weyl CCR.

**Corollary B.1.** *For $x \in \mathbb{R}^N$, $\xi = (p,q) \in \Xi$ and $\psi \in \mathcal{H}$ we define the Weyl CCR representation:*

$$(W((p,q))(\psi))(x) =: W(\xi)(\psi)(x) = e^{\frac{ip\cdot q}{2} + ip\cdot x}\psi(x+q). \tag{B.1}$$

*This representation is well defined.*

*Proof.* We have to show that the given $W(\xi)$ is a unitary operator for every $\xi$ and that $W$ obeys the Weyl CCR. Starting with the commutation relations, one obtains

$$\begin{aligned}
W(\xi)W(\eta)(\psi)(x) = W(p,q)W(p',q')(\psi)(x) &= W(\xi)e^{\frac{ip'\cdot q'}{2}+ip'\cdot x}\,\psi(x+q') \\
&= e^{\frac{ip'\cdot q'}{2}+ip'\cdot x}\,e^{\frac{ip\cdot q}{2}+ip\cdot x}\,\psi(x+q'+q) \\
&= e^{\frac{ip\cdot q'}{2}-\frac{iq\cdot p'}{2}}\,e^{\frac{i(p+p')\cdot(q+q')}{2}+i(p+p')\cdot x}\,\psi(x+q+q') \\
&= e^{\frac{i(p\cdot q'-q\cdot p')}{2}}\,W(\xi+\eta)(\psi)(x).
\end{aligned}$$

Its adjoint can be calculated by

$$\begin{aligned}
\langle\psi|W(\xi)(\varphi)\rangle_{\mathcal{H}} &= \int_{\mathbb{R}^N} \mathrm{d}^N x\;\overline{\psi(x)}\,W(\xi)(\varphi)(x) = \int_{\mathbb{R}^N} \mathrm{d}^N x\;\overline{\psi(x)}\,e^{\frac{ip\cdot q}{2}+ip\cdot x}\varphi(x+q) \\
&= \int_{\mathbb{R}^N} \mathrm{d}^N \tilde{x}\;\overline{\psi(\tilde{x}-q)}\,e^{\frac{ip\cdot q}{2}+ip\cdot(\tilde{x}-q)}\varphi(\tilde{x}) \\
&= \int_{\mathbb{R}^N} \mathrm{d}^N \tilde{x}\;\overline{\psi(\tilde{x}-q)}\,e^{-\frac{ip\cdot q}{2}+ip\cdot\tilde{x}}\varphi(\tilde{x}) \\
&= \int_{\mathbb{R}^N} \mathrm{d}^N \tilde{x}\;\overline{\psi(\tilde{x}-q)\,e^{\frac{ip\cdot q}{2}-ip\cdot\tilde{x}}}\varphi(\tilde{x}) =: \left\langle W(\xi)^\dagger(\psi)\middle|\varphi\right\rangle_{\mathcal{H}}.
\end{aligned}$$

Therefore we have $W(p,q)^\dagger(\psi) = W(-p,-q)(\psi)$ and hence the given $W$ obeys the Weyl CCR as in equation (2.26). □

We start with a very important theorem, which will be used quite often.

**Theorem B.2.** *Let $\rho_1,\rho_2 \in \mathfrak{T}(\mathcal{H})$ trace-class operators. Then the function*

$$\xi \longmapsto \operatorname{tr}\Big(\rho_1 W(\xi)\rho_2 W(\xi)^\dagger\Big) \tag{B.2}$$

*is integrable and $W$ is an isometry in the sense that*

$$\int_{\Xi} \mathrm{d}^N\xi\;\operatorname{tr}\Big(\rho_1 W(\xi)\rho_2 W(\xi)^\dagger\Big) = \operatorname{tr}(\rho_1)\operatorname{tr}(\rho_2). \tag{B.3}$$

*Proof.* Since $\rho_1, \rho_2 \in \mathfrak{T}(\mathcal{H})$ are trace class and therefore especially compact operators, we can rewrite them via their spectral decomposition as

$$\rho_i = \sum_{n=1}^{\infty} \lambda_n P_n \quad \text{with} \quad i = 1, 2,$$

where the point spectrum $\sigma_p(\rho_i) = \{\lambda_1, \lambda_2, \dots\}$ consists of, at most, countably infinite elements with zero as only possible accumulation point. The sum converges in norm. Since the trace is linear, we just need to look at the case where $\rho_i = P_{\psi_i} = |\psi_i\rangle\langle\psi_i|$ are one-dimensional projectors with $\|\psi_i\|_{\mathcal{H}} = 1$. The integral (B.3) then reduces to

$$\begin{aligned}
\int_\Xi \mathrm{d}^N\xi \ \mathrm{tr}\Big(\rho_1 W(\xi)\rho_2 W(\xi)^\dagger\Big) &= \int_\Xi \mathrm{d}^N\xi \ \sum_i \Big\langle \psi_i \Big| P_{\psi_1} W(\xi) P_{\psi_2} W(\xi)^\dagger \psi_i \Big\rangle_{\mathcal{H}} \\
&= \int_\Xi \mathrm{d}^N\xi \ \langle \psi_1 | W(\xi)\psi_2\rangle \, \langle \psi_2 | W(-\xi)\psi_1\rangle_{\mathcal{H}} \\
&= \int_\Xi \mathrm{d}^N\xi \ \langle \psi_1 | W(\xi)\psi_2\rangle_{\mathcal{H}} \, \overline{\langle \psi_1 | W(\xi)\psi_2\rangle}_{\mathcal{H}} \\
&= \int_\Xi \mathrm{d}^N\xi \, |\langle \psi_1 | W(\xi)\psi_2\rangle \, \mathcal{H}|^2.
\end{aligned}$$

We define $\varphi_q(x) := \overline{\psi_1(x)}\psi_2(x+q)$ and see that $\varphi_q(x) \in L^1(\mathbb{R}^N, \mathrm{d}^N x)$. Furthermore we have

$$\begin{aligned}
\langle \psi_1 | W((p,q))\psi_2\rangle_{\mathcal{H}} &= \int_{\mathbb{R}^N} \mathrm{d}^N x \, \overline{\psi_1(x)} \, W((p,q))(\psi_2)(x) \\
&= e^{\frac{ip\cdot q}{2}} \int_{\mathbb{R}^N} \mathrm{d}^N x \, \overline{\psi_1(x)} \, e^{ip\cdot x} \psi_2(x+q) \\
&= e^{\frac{ip\cdot q}{2}} \int_{\mathbb{R}^N} \mathrm{d}^N x \, \varphi_q(x) e^{ip\cdot x} = (2\pi)^{N/2} e^{\frac{ip\cdot q}{2}} \, \widehat{\varphi}_q(p),
\end{aligned} \tag{B.4}$$

where $\widehat{\varphi}_q(p)$ denotes the Fourier transform of $\varphi_q(x)$. Hence

$$\begin{aligned}
(\text{B.3}) &= \int\limits_{\Xi} \mathrm{d}^N\xi \, |\langle\psi_1|W(\xi)\psi_2\rangle_{\mathcal{H}}|^2 \overset{(\text{B.4})}{=} \int\limits_{\Xi} \frac{\mathrm{d}^N q \, \mathrm{d}^N p}{(2\pi)^N} \left|(2\pi)^{N/2} e^{\frac{ip\cdot q}{2}} \, \widehat{\varphi}_q(p)\right|^2 \\
&= \int\limits_{\Xi} \mathrm{d}^N p \, \mathrm{d}^N q \left|\widehat{\varphi}_q(p)\right|^2 = \int\limits_{\mathbb{R}^N \times \mathbb{R}^N} \mathrm{d}^N x \, \mathrm{d}^N q \left|\varphi_q(x)\right|^2 \\
&= \int\limits_{\mathbb{R}^{2N}} \mathrm{d}^N x \, \mathrm{d}^N q \left|\overline{\psi_1(x)}\psi_2(x+q)\right|^2 \\
&= \int\limits_{\mathbb{R}^{2N}} \mathrm{d}^N x \, \mathrm{d}^N q \left|\overline{\psi_1(x)}\right|^2 |\psi_2(x+q)|^2 = \|\psi_1\|^2_{\mathcal{H}} \, \|\psi_2\|^2_{\mathcal{H}} = 1,
\end{aligned}$$

where we have used that the Fourier transformation is an isometry and therefore

$$\int\limits_{\mathbb{R}^N_{\geq 0}} \mathrm{d}^N p \left|\widehat{\varphi}_q(p)\right| = \int\limits_{\mathbb{R}^N} \mathrm{d}^N x \left|\varphi_q(x)\right| \quad \forall\, q \in \mathbb{R}^N.$$

□

In so-called "hybrid systems" one is going to encounter spaces of the form $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, where $\mathcal{A}_0$ describes the dynamics of classical physics and $\mathcal{A}_1$ of quantum mechanics. A typical example would be the space of observables $L^\infty(\Xi, \mathrm{d}^N\xi) \oplus \mathcal{B}(\mathcal{H}) =: \mathcal{R}^\infty = \mathcal{R}^\infty_0 \oplus \mathcal{R}^\infty_1$. At first we need to define some operations on those spaces.

**Definition B.3.** *Denote the set of complex valued functions on $\{0,1\} \times \Xi$ as $\mathscr{C} = \mathscr{C}_0 \oplus \mathscr{C}_1$ and the subspace of square integrable functions will be denoted as $\mathscr{C}^2 = \mathscr{C}^2_0 \oplus \mathscr{C}^2_1$ with the product*[1]

$$(fg)_i(\xi) = \sum_{k \in \mathbb{Z}_2} f_k(\xi) g_{k+1}(\xi). \tag{B.5}$$

*The **Fourier transformation** $\mathscr{F}$ will be defined separately as $\mathscr{F} = \mathscr{F}_0 \oplus \mathscr{F}_1 : \mathcal{R}^1 \longrightarrow \mathscr{C}$ with*

$$\mathscr{F}_0 : L^1(\Xi, \mathrm{d}\xi) \longrightarrow \mathscr{C}_0, \qquad (\mathscr{F}_0 f)(\xi) = \int \mathrm{d}\eta \, e^{i\sigma(\xi,\eta)} f(\eta) \tag{B.6}$$

[1] Whenever we encounter the space $\mathbb{Z}_2$ we equip it with the addition mod 2.

$$\mathscr{F}_1 : \mathfrak{T}(\mathcal{H}) \longrightarrow \mathscr{C}_1, \qquad (\mathscr{F}_1 A)(\xi) = \operatorname{tr}(W(\xi)A). \tag{B.7}$$

This Fourier transform obeys most of the usual notions one would expect, for example its compatibility with the convolution, i.e. $\mathscr{F}(A * B) = \mathscr{F}(A)\mathscr{F}(B)$. We need to mention one very important theorem, since it will be quite relevant in the ongoing discussion.

**Theorem B.4** (Bochner)**.** *Let $f : \{0,1\} \times \Xi \longrightarrow \mathbb{C}$ be a function. $f$ is the Fourier transform of some positive $T \in \mathcal{R}^1$ iff $f$ is continuous and for all $\xi_1, \ldots, \xi_n \in \Xi$ and $z_1, \ldots, z_n \in \mathbb{C}$ the two following identities hold:*

*i)* $\sum_{j,k=1}^{n} f_0(\xi_j - \xi_k)\overline{z}_j z_k \geq 0.$

*ii)* $\sum_{j,k=1}^{n} f_1(\xi_j - \xi_k)e^{\frac{i}{2}\sigma(\xi_j,\xi_k)}\overline{z}_j z_k \geq 0.$

*Proof.* A proof for this theorem may be found in [Luk70][Theorem 4.2.1., p. 71]. □

We can now focus on special examples of channels and analyze their behavior acting on Weyl operators.

## B.1 Gaussian Quantum Channels

Let $T : \mathfrak{T}(\mathcal{H}) \longrightarrow \mathfrak{T}(\mathcal{H})$ be completely positive channel and **covariant** w.r.t. the action $\alpha_\xi$, i.e. $\alpha_\xi T(\rho) = T(\alpha_\xi \rho)\ \forall \xi \in \Xi$. The notion of covariance of semigroups and their application to quantum channels is due to Holevo [Hol95a]. The dual space of $\mathfrak{T}(\mathcal{H})$ is $\mathcal{B}(\mathcal{H})$ and $T^\dagger : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ is completely positive and covariant, too. If we consider the "translation" action of $\alpha_\eta$ on the adjoint, we see

$$\alpha_\eta T^\dagger(W(\xi)) = T^\dagger(\alpha_\eta W(\xi)) = T^\dagger\left(W(\eta)W(\xi)W(\eta)^\dagger\right) = e^{i\sigma(\eta,\xi)}T^\dagger(W(\xi)). \tag{B.8}$$

The complex phase $e^{i\sigma(\eta,\xi)}$ is the eigenvalue to the translation. Furthermore we can conclude:

**Corollary B.5.** *The following equivalence holds for all $\eta \in \Xi$ and $A \in \mathcal{B}(\mathcal{H})$*

$$\alpha_\eta(A) = e^{i\sigma(\eta,\xi)}A \iff A = \lambda W(\xi),\ \lambda \in \mathbb{C}. \tag{B.9}$$

*Proof.* The statement is already shown for $A$ being a Weyl operator. Let $A$ be an arbitrary operator and $\widetilde{A} = AW(\xi)^\dagger$, then $\alpha_\xi \widetilde{A} = \widetilde{A}$ and $\widetilde{A}W(\xi) = W(\xi)\widetilde{A}$. But that can only be the case when $\widetilde{A} = \lambda \mathbb{1}$ for a $\lambda \in \mathbb{C}$, which proves the statement. □

This corollary implies that $T^\dagger(W(\xi)) = \lambda(\xi)W(\xi)$ and we see:

**Proposition B.6.** *The following equation for $T$ holds*

$$\mathscr{F}T(\rho)(\xi) = \lambda(\xi)(\mathscr{F}\rho)(\xi), \quad \rho \in \mathfrak{T}(\mathcal{H}). \tag{B.10}$$

*Proof.* We start by letting the Fourier transformation act on both sides of $T^\dagger(W(\xi)) = \lambda(\xi)W(\xi)$ and use its linearity:

$$\left(\mathscr{F}T^\dagger(W(\xi))\right)(\eta) = \lambda(\xi)(\mathscr{F}W(\xi))(\eta).$$

Analyzing both sides separately and using $\operatorname{tr}\left(\rho T^\dagger(A)\right) = \operatorname{tr}(T(\rho)A)$ and the cyclic property of the trace, we obtain for the LHS

$$\begin{aligned}\left(\mathscr{F}T^\dagger(W(\xi))\right)(\eta) &= \operatorname{tr}\left(W(\eta)T^\dagger(W(\xi))\right) = \operatorname{tr}(T(W(\eta))W(\xi)) \\ &= \operatorname{tr}(W(\xi)T(W(\eta))) = (\mathscr{F}T(W(\eta)))(\xi).\end{aligned}$$

And similarly for the LHS

$$\begin{aligned}\lambda(\xi)(\mathscr{F}W(\xi))(\eta) &= \lambda(\xi)\operatorname{tr}(W(\eta)W(\xi)) = \lambda(\xi)\operatorname{tr}(W(\xi)W(\eta)) \\ &= \lambda(\xi)(\mathscr{F}W(\eta))(\xi).\end{aligned}$$

Comparing both sides and denoting $W(\eta) = \rho$ proves the statement. □

We can use Bochner's theorem to get necessary criteria for $\lambda(\xi)$ such that $T$ remains completely positive. One example of an allowed choice of $\lambda$ would be the Gaussian function $\lambda(\xi) = e^{-\xi Y \xi}$ with a linear operator $Y : \Xi \longrightarrow \Xi$. If we allow the argument to be transformed, we'll get a more general notion:

$$T^\dagger(W(\xi)) = e^{-\xi Y \xi}\, W(X\xi), \quad \text{for} \quad X, Y : \Xi \longrightarrow \Xi \text{ linear.} \tag{B.11}$$

A channel $T$ that admits such a $\lambda$ will be called a **Gaussian channel**. Using the Stinespring theorem, we'll obtain necessary criteria for the complete positivity of $T^\dagger$:

$$\sum_{j,k\in\mathbb{Z}_2} B_j^\dagger\, T^\dagger\left(A_j^\dagger A_k\right) B_k \geq 0. \tag{B.12}$$

For the $A, B$ we choose Weyl operators, s.t.

$$\begin{aligned}
&\sum_{j,k\in\mathbb{Z}_2} W(\eta_j)^\dagger\, T^\dagger\left(W(\xi_j)^\dagger W(\xi_k)\right) W(\eta_k) \\
&= \sum_{j,k\in\mathbb{Z}_2} e^{\frac{i}{2}\sigma(-\xi_j,\xi_k)}\, W(\eta_j)^\dagger T^\dagger\left(W(-\xi_j+\xi_k)\right) W(\eta_k) \\
&= \sum_{j,k\in\mathbb{Z}_2} e^{-\frac{i}{2}\sigma(\xi_j,\xi_k)-(\xi_k-\xi_j)Y(\xi_k-\xi_j)}\, W(\eta_j)^\dagger W(X(\xi_k-\xi_j))W(\eta_k) \\
&= \sum_{j,k\in\mathbb{Z}_2} e^{-(\xi_k-\xi_j)Y(\xi_k-\xi_j)-\frac{i}{2}\sigma(\xi_j,\xi_k)+\frac{i}{2}\sigma(X\xi_j,X\xi_k)}\, W(\eta_j)^\dagger W(-X\xi_j)W(X\xi_k)W(\eta_k),
\end{aligned}$$

where the last equation comes from the CCR, since

$$W(X\xi_k - X\xi_j) = e^{\frac{i}{2}\sigma(X\xi_j,X\xi_k)} W(-X\xi_j)W(X\xi_k).$$

Using $W(-X\xi_j) = W\left(X\xi_j\right)^\dagger$ and defining $S_i = W(X\xi_i)W(\eta_i)$ we get

$$W(X\xi_k - X\xi_j) = \sum_{j,k\in\mathbb{Z}_2} e^{-\xi_j Y\xi_j - \xi_k Y\xi_k + 2\xi_j Y\xi_k - \frac{i}{2}\sigma(\xi_j,\xi_k)+\frac{i}{2}\sigma(X\xi_j,X\xi_k)} S_j^\dagger S_k.$$

Since the $S_j^\dagger S_k$ term is positive it is a sufficient condition for semi definite positivity that the exponent is positive. Writing the symplectic form in a basis, we denote the symplectic form as its matrix representation $\sigma$. Hence we require the following matrix to be non negative:

$$\exp\left(-\xi_j Y\xi_j - \xi_k Y\xi_k + 2\xi_j Y\xi_k - \frac{i}{2}\xi_j\cdot\sigma\xi_k + \frac{i}{2}X\xi_j\cdot\sigma X\xi_k\right) \geq 0 \quad \forall j,k. \tag{B.13}$$

Since the first two summands are Gaussian and therefore positive, we can state the sufficient condition

$$\exp\left(\xi_j\left(2Y-\frac{i}{2}\sigma+\frac{i}{2}X^T\sigma X\right)\xi_k\right)\geq 0 \implies 2Y-\frac{i}{2}\left(\sigma-X^T\sigma X\right)\geq 0. \quad \text{(B.14)}$$

## Composition of Gaussian Channels

Consider two Gaussian channels $T_1, T_2$ in the Schrödinger picture, i.e. $T_1, T_2 : \mathfrak{T}(\mathcal{H}) \longrightarrow \mathfrak{T}(\mathcal{H})$. The concatenation of those channels is again a Gaussian channel, since

$$\begin{aligned} T_1T_2(W(\xi)) &= e^{-\xi Y_2\xi}\, T_1(W(X_2\xi)) = e^{-\xi Y_2\xi-(X_2\xi)Y_1(X_2\xi)}\, W(X_1X_2\xi) \\ &=: e^{-\xi\widetilde{Y}\xi}\, W(\widetilde{X}\xi), \end{aligned} \quad \text{(B.15)}$$

with $\widetilde{X} = X_1X_2$ and $\widetilde{Y} = Y_2 + X_2^TY_1X_2$. If $X$ is derived via a semigroup, i.e. $X_t = \exp\left(t\dot{X}\right)$ and $X_{t+s} = X_tX_s$ with generator $\dot{X} = \frac{\mathrm{d}}{\mathrm{d}t}X_t\Big|_{t=0}$, we get

$$\begin{aligned} Y_{t+s} = Y_t + X_t^TY_sX_t, \qquad \dot{Y} &= \frac{\mathrm{d}}{\mathrm{d}t}Y_t\Big|_{t=0}, \qquad Y_0 = 0 \\ \implies \dot{Y}_t = X_t^T\dot{Y}X_t, \qquad Y_t &= \int_0^t \mathrm{d}s\, X_s^T\dot{Y}X_s. \end{aligned}$$

When this is the case, we denote the channel as $T_t(W(\xi)) = e^{-\xi Y_t\xi}\, W(X_t\xi)$. The c.p. condition becomes

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(2Y_t-\frac{i}{2}\left(\sigma-X_t^T\sigma X_t\right)\right)\Big|_{t=0} \\ &\quad = 2\dot{Y}+\frac{i}{2}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}X_t^T\right)\sigma X_t + X_t^T\sigma\left(\frac{\mathrm{d}}{\mathrm{d}t}X_t\right)\right]\Big|_{t=0} \\ &\quad = 2\dot{Y}+\frac{i}{2}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}e^{t\dot{X}^T}\right)\sigma e^{t\dot{X}} + e^{t\dot{X}^T}\sigma\left(\frac{\mathrm{d}}{\mathrm{d}t}e^{t\dot{X}}\right)\right]\Big|_{t=0} \\ &\quad = 2\dot{Y}+\frac{i}{2}\left(\dot{X}^T\sigma+\sigma\dot{X}\right)\geq 0. \end{aligned} \quad \text{(B.16)}$$

**Theorem B.7.** *There exists a correspondence between pairs* $(\dot{X}, \dot{Y})$ *and* $(M, H)$ *of* $2n \times 2n$ *matrices, s.t.*

- $(\dot{X}, \dot{Y})$ *real and* $2\dot{Y} + \frac{i}{2}\left(\dot{X}^T\sigma + \sigma\dot{X}\right) \geq 0$.
- $M$ *complex valued,* $M \geq 0$, $M^* = M^T$ *and* $H$ *real,* $H = H^T$.

*The correspondence is given by*

$$\dot{X} = (\operatorname{Im} M + H)\sigma \tag{B.17}$$

$$\dot{Y} = -\frac{1}{2}\sigma(\operatorname{Re} M)\sigma. \tag{B.18}$$

*Proof.* Since $\sigma$ is the matrix representation of a symplectic form it satisfies $\sigma^T = -\sigma = \sigma^{-1}$. Using this and the assumptions from the theorem one can compute the positivity condition (B.16) as

$$\begin{aligned}
\text{(B.16)} &= 2\dot{Y} + \frac{i}{2}\left(\dot{X}^T\sigma + \sigma\dot{X}\right)\\
&= -\sigma(\operatorname{Re} M)\sigma + \frac{i}{2}\left(-\sigma\left(\operatorname{Im} M + H\right)^T\sigma + \sigma\left(\operatorname{Im} M + H\right)\sigma\right)\\
&= -\sigma(\operatorname{Re} M)\sigma + \frac{i}{2}\left(-\sigma(\operatorname{Im} M^*)\sigma + \sigma(\operatorname{Im} M)\sigma - \sigma H\sigma + \sigma H\sigma\right)\\
&= \sigma(\operatorname{Re} M)\sigma^T + \frac{i}{2}\sigma(\operatorname{Im} M^*)\sigma^T - \frac{i}{2}\sigma(\operatorname{Im} M)\sigma^T\\
&= \sigma\left(\operatorname{Re} M - i\operatorname{Im} M\right)\sigma^T = \sigma M^*\sigma^T \geq 0 \iff M \geq 0.
\end{aligned}$$

□

**Theorem B.8.** *The aforementioned matrices* $(M, H)$ *generate the same dynamical semigroup as* $(\dot{X}, \dot{Y})$, *i.e.*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}T_t\right|_{t=0} = \mathcal{L}^{q.f.} = \mathcal{L}^{Lindblad}. \tag{B.19}$$

The proof is outlined in [Sie].

**Theorem B.9.** *The Lindbladian of this Gaussian evolution, i.e.*

$$\mathcal{L}(W) = \frac{1}{2}\sum_{\alpha,\beta} M_{\alpha\beta}\left([R_\alpha, W]R_\beta + R_\alpha[W, R_\beta]\right) - \frac{i}{2}\sum_{\alpha,\beta} H_{\alpha\beta}[R_\alpha R_\beta, W], \tag{B.20}$$

*is of standard form*

$$\mathcal{L}(W) = K^\dagger W + WK + \sum_{\alpha,\beta} M_{\alpha\beta} R_\alpha W R_\beta, \tag{B.21}$$

*with* $K = -\frac{1}{2}\sum_{\alpha,\beta}(M - iH)_{\alpha\beta}R_\alpha R_\beta$ *and operators* $\{R_\alpha\}_{\alpha=1,\dots,2n}$ *which are defined as*

$$(R_1, \dots, R_{2n}) = (P_1, \dots, P_n, Q_1, \dots, Q_n).$$

*Therefore the Weyl operators are (up to a unitary representation) of the form* $W(\xi) = e^{i\xi\cdot R} = e^{i\sum_\alpha \xi_\alpha R_\alpha}$.

*Proof.* We prove this theorem via a straightforward calculation. Plugging $K$ into the standard form and reorder the terms we have

$$\begin{aligned}\mathcal{L}(W) = &-\frac{1}{2}\sum_{\alpha,\beta} R_\beta R_\alpha (M^*_{\alpha\beta} + iH^*_{\alpha\beta})W - \frac{1}{2}W\sum_{\alpha,\beta}(M_{\alpha\beta} - iH_{\alpha\beta})R_\alpha R_\beta \\ &+ \sum_{\alpha,\beta} M_{\alpha\beta} R_\alpha W R_\beta \\ = &\sum_{\alpha,\beta} -\frac{1}{2}M_{\beta\alpha}R_\beta R_\alpha W - \frac{i}{2}H_{\alpha\beta}R_\beta R_\alpha W - \frac{1}{2}M_{\alpha\beta}WR_\alpha R_\beta \\ &+ \frac{i}{2}H_{\alpha\beta}WR_\alpha R_\beta + M_{\alpha\beta}R_\alpha W R_\beta.\end{aligned}$$

Comparing this with

$$(\text{B.20}) = \frac{1}{2}\sum_{\alpha,\beta} M_{\alpha\beta}(R_\alpha W R_\beta - WR_\alpha R_\beta + R_\alpha W R_\beta - R_\alpha R_\beta W)$$

$$
\begin{aligned}
&- \frac{i}{2} \sum_{\alpha,\beta} H_{\alpha\beta}(R_\alpha R_\beta W - W R_\alpha R_\beta) \\
= \sum_{\alpha,\beta} &- \frac{1}{2} M_{\alpha\beta} W R_\alpha R_\beta - \frac{1}{2} M_{\alpha\beta} R_\alpha R_\beta W - \frac{i}{2} H_{\alpha\beta} R_\alpha R_\beta W \\
&+ \frac{i}{2} H_{\alpha\beta} W R_\alpha R_\beta + M_{\alpha\beta} R_\alpha W R_\beta,
\end{aligned}
$$

we obtain equality. □

The biggest merit of this analysis is, that the standard form of the Lindbladian gives us insight about the interpretation of open quantum systems. The Lindbladian (B.21) clearly splits into two parts:

i) A **dissipator** given by $\mathcal{Z}(W) := K^\dagger W + WK$.

ii) A **jump map** given by $\mathcal{J}(W) := \sum_{\alpha,\beta} M_{\alpha\beta} R_\alpha W R_\beta$.

Evolutions of the dissipator belong to evolutions of the form $W \longrightarrow U(t)^\dagger W U(t)$, where $U(t)$ is the semigroup generated by $K$, i.e. $U(t) = \exp(tK)$. However, looking at the definition of $K$, we see that it contains itself two contributions.
The first part of the evolutions is unitary, since it is generated by $iH$ and corresponds to a free evolution. Secondly there is the manifestly negative operator in $K$ "destroying" information, in the sense that there are pure states $|\psi\rangle \in \mathcal{H}$ with $\|U(t)\,|\psi\rangle\| < 1$ for $t > 0$.
Evolutions generated by the jump map can be interpreted as quantum events intertwined with a absorptive evolution as outlined in [Neu15][Section 3.1.2.].

Those Gaussian channels yield mathematically rich structure and should be an interesting toy model for the continuous measurement formalism explained in this thesis, especially because a lot is known about quantum systems with Lindbladians in standard form.

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Thank you!

# List of Figures

# Nomenclature

**Algebraic Constructions**

| | |
|---|---|
| $(\Xi, \sigma)$ | Symplectic Space |
| $\alpha^*$ | Complex conjugate of $\alpha \in \mathbb{C}$ |
| $f\text{-}\varinjlim_{i \in I} X_i$ | Inductive limit of the inductive system $(X_i, f_{ij})$ |
| $f\text{-}\varprojlim_{i \in I} X_i$ | Inverse limit of the direct system $(X_i, f_{ij})$ |
| $\varinjlim_{i \in I}$ | Net limit w.r.t. the directed set $I$ |
| $\lim_{\Xi \gg \Theta}$ | Discrete comparison limit |
| $\mathcal{A}'$ | Topological dual of the vector space $\mathcal{A}$ |
| $\mathcal{A}_+$ | Positive elements of the Banach algebra $\mathcal{A}$ |
| $\\|\cdot\\|_{\mathrm{op}}$ | Operator norm |
| $\sigma(A)$ | Spectrum of the Banach algebra A |
| $V_{\mathbb{C}}$ | Complexification of the real vector space $V$ |
| $\mathrm{CCR}(\Xi, \sigma)$ | CCR C*-algebra over the symplectic space $(\Xi, \sigma)$ |

**Continuous Measurement Constructions**

| | |
|---|---|
| $\mathfrak{M}_i^{\Xi}(\lvert\lambda\rangle)$ | Averaging the function $\lvert\lambda\rangle$ over the $\Xi\rvert_i$ subinterval |
| $\lvert e(\lambda)\rangle$ | Limit exponential vector |
| $\lvert e_\Theta(\lambda)\rangle$ | Exponential vector w.r.t. the interval decomposition $\Theta$ |
| $\mathcal{K}_\Theta$ | The discrete dilation space w.r.t. the interval decomposition $\Theta$ |
| $\mathcal{K}_{[0,T]}$ | Limit dilation space |

| | |
|---|---|
| $\overline{\lambda}_i$ | Average of the function $\lambda$ over the interval $[t_{i-1}, t_i)$ |
| $W(\|\lambda\rangle, U)$ | Continuous Weyl operator |
| $W_\Theta(\|\lambda\rangle)$ | Discrete Weyl operator without unitary rotations |

**Function Spaces**

| | |
|---|---|
| $\mathscr{S}([0,T],\mathcal{K})$ | Schwartz functions on $\mathcal{K}$ |
| $\mathrm{Step}_I([0,T],\mathcal{K})$ | Step functions on $\mathcal{K}$ |
| $C([0,T],\mathcal{K})$ | Continuous functions on $\mathcal{K}$ |
| $C_c^\infty([0,T],\mathcal{K})$ | Smooth functions with compact support on $\mathcal{K}$ |
| $L^0(X,Y)$ | Measurable functions from the measureable spaces $X$ to $Y$ |
| $L^p(S,\Omega,\mu,X)$ | p-times Bochner integrable functions over a measure space $(S,\Omega,\mu)$ on a Banach space $X$ |

**Number Sets**

| | |
|---|---|
| $\mathbb{N}$ | Natural numbers |
| $\mathbb{N}_0$ | Natural numbers including zero |
| $\mathbb{Z}$ | Integers |
| $\mathbb{Q}$ | Rational numbers |
| $\mathbb{R}$ | Real numbers |
| $\mathbb{C}$ | Complex numbers |
| $\mathfrak{Z}([0,T])$ | Interval decompositions over $[0,T]$ |
| $I(\Theta)$ | Index set of the interval decomposition $\Theta \in \mathfrak{Z}([0,T])$ |
| $\#I(\Theta)$ | Cardinality of the set $I(\Theta)$ |

**Operator Spaces**

| | |
|---|---|
| $\mathcal{B}(X,Y)$ | Bounded linear operators from $X$ to $Y$ |
| $\mathfrak{CP}(\mathcal{A},\mathcal{B})$ | Completely positive maps between C*-algebras $\mathcal{A},\mathcal{B}$ |
| $\mathfrak{T}(\mathcal{H})$ | Trace class operators on $\mathcal{H}$ |
| $K(\mathcal{H})$ | Compact operators on $\mathcal{H}$ |
| $L(X,Y)$ | Linear operators from $X$ to $Y$ |

**Stochastic Constructions**

| | |
|---|---|
| $(\Gamma F)$ | Fock space observable corresponding to one-particle POVM $F$ |
| $\Gamma_\Theta$ | Second quantization operator functor |
| $\mathcal{CM}(X, \mathfrak{X})$ | Counting measures on a measurable space |
| $\mathfrak{P}_\Theta$ | Point process w.r.t the interval decomposition $\Theta$ |
| $\mu[f]$ | Integral over the function $f$ with the measure $\mu$ |
| $\Phi_\Theta(\|\lambda\rangle)$ | Second quantized field operator |
| $\Upsilon_*(\mu)$ | Pushforward of the measure $\mu$ along $\Upsilon$ |
| $\widehat{C}(f)$ | Generating function of the factorial moments |
| $C(f)$ | Characteristic function of a stochastic process |

**Abbrevations**

| | |
|---|---|
| CCR | Canonical Anticommutation Relations |
| CCR | Canonical Commutation Relations |
| cMPS | Continuous Matrix Product States |
| FCS | Finitely Correlated States |
| iff | if and only if |
| MPS | Matrix Product States |
| POVM | Positive Operator Valued Measure |
| PVM | Projection Valued Measure |
| t.r.o. | to relevant order in $\tau$ |

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# Declaration of Authorship

I, Michael Werner, declare that this thesis, titled “One-dimensional field theories arising from continuous measurement”, and the work presented in it are my own. I confirm that:

- This work was done wholly by myself.
- I acknowledged all references and all sources of help.
- This work in this or similar form has not been submitted to any other institution.
- Whenever I have used published work of others, this is always clearly attributed.

Signed: ______________________________

Hannover, 06.08.2019